D6BMT2301 **Reg.No............................**
Name: ....

SIXTH SEMESTER B. Sc. DEGREE EXAMINATION, APRIL 2026
(Regular/Improvement/Supplementary)
MATHEMATICS
GMAT6B10T - ADVANCED REAL ANALYSIS

Time: 2.5 Hours **Maximum Marks: 80**

SECTION A: Answer the following questions. Each carries two marks. (Ceiling 25 marks)

1. Define a continuous function and give an example with justification.
2. Define absolute maximum and absolute minimum. State the Maximum-Minimum Theorem.
3. If f and g are increasing functions on an interval $I \subseteq \mathbb{R}$, show that $f+g$ is an increasing function on I.
4. Show that every constant function on the closed interval $[a,b]$ is in $R[a,b]$.
5. If $f:[a,b]\to\mathbb{R}$ is continuous on $[a,b]$, then show that $f\in R[a,b]$.
6. If $f:\mathbb{R}\to\mathbb{R}$ is continuous and $C>0$, define $G:\mathbb{R}\to\mathbb{R}$ by
$$G(x)=\int_{x-C}^{x+C} f(t)dt$$
Show that G is differentiable on $\mathbb{R}$ and find $G'(x)$.
7. Consider the function H defined by: $H(x)=x+1$ for $x\in[0,1]$ rational, and $H(x)=0$ for $x\in[0,1]$ irrational. Show that H is not Riemann integrable on $[0,1]$.
8. State the general form of integration by parts for the Riemann integral and Taylor's theorem with the remainder.
9. Define pointwise convergence and uniform convergence of a sequence of functions. Give an example for each.
10. Show that $\lim_{n\to\infty}\frac{x}{x+n}=0, \quad \forall x\in\mathbb{R}, x\geq 0$.
11. Define the following terms for a sequence of functions $\{f_n\}$ defined on a subset D of $\mathbb{R}$ with values in $\mathbb{R}$:
    a. $\sum f_n$ is absolutely convergent on D.
    b. $\sum f_n$ is uniformly convergent on D.
12. State the Fundamental Theorem of Integral Calculus.
13. Define Cauchy Principal Value.
14. State the Absolute Root Test for convergence of a series. **PTO**

15. Define nice function.

SECTION B: Answer the following questions. Each carries five marks.

(Ceiling 35 marks)

16. Let $k > 0$ and $f : \mathbb{R} \to \mathbb{R}$ satisfy the condition $|f(x) - f(y)| \leq k|x-y|$ for all $x, y \in \mathbb{R}$. Show that f is continuous at every point $c \in \mathbb{R}$.

17. Prove that a function f is uniformly continuous on the open interval (a, b) if and only if it can be extended to a function $\tilde{f}$ on the closed interval $[a, b]$ such that $\tilde{f}$ is continuous on $[a, b]$.

18. If $f \in R[a, b]$, then show that f is bounded on the closed interval $[a, b]$.

19. Prove that the indefinite integral F defined by $F(x) = \int_a^x f$ for all $x \in [a, b]$ is continuous on $[a, b]$. If $|f(x)| \leq M$ for all $x \in [a, b]$, then show that $|F(s) - F(t)| \leq M|s-t|$ for all $s, t \in [a, b]$.

20. Let (f_n) be a sequence of continuous functions on a set $A \subset \mathbb{R}$ and suppose that the sequence (f_n) converges uniformly to a function f from A to $\mathbb{R}$. Then prove that f is continuous on A.

21. Prove that for $-\infty \leq a < 0 < b \leq \infty$, the integral $\int_a^b \frac{1}{x}\, dx$ does not exist.

22. Find

    a. $\int_0^9 \frac{x}{(x-3)^2} dx$

    b. $PV \int_0^9 \frac{x}{(x-3)^2} dx$

23. Prove that for all $q \leq -1$, $\int_0^1 x^q e^{-x} dx$ diverges

SECTION C: Answer any two questions. Each carries ten marks.

24. (a) State and prove the Boundedness Theorem.

    (b) State and prove the Preservation of Intervals Theorem.

25. State and prove the Cauchy criterion for Riemann integrability.

26. Let $\{f_n\}$ be a sequence of functions in $R[a, b]$ and suppose that f_n converges uniformly on $[a, b]$ to a function f. Prove that $f \in R[a, b]$ and that

$$\int_a^b f(x)dx = \lim_{n\to\infty} \int_a^b f_n(x)dx.$$

27. Evaluate the double integral:

$$\iint_{\overline{D_(0,a)}} e^{-(x^2+y^2)} dxdy$$

where $\overline{D(0, a)} = \{(x, y) | x^2 + y^2 \leq a^2\}$ is the closed disc centered at $(0, 0)$ with radius $a > 0$.

(2 × 10 = 20 Marks)