D6BMT2203 **(Pages: 3)** **Reg.No...............................**

Name: ....

SIXTH SEMESTER B.Sc. DEGREE EXAMINATION, APRIL 2025

(Regular/Improvement/Supplementary)

MATHEMATICS

GMAT6B12T: LINEAR ALGEBRA

Time: 2.5 Hours **Maximum Marks: 80**

SECTION A: Answer the following questions. Each carries two marks.

(Ceiling 25 marks)

1. Verify that $\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$ is an eigenvector of the 3×3 matrix $A = \begin{pmatrix} 0 & -1 & -3 \\ 2 & 3 & 3 \\ -2 & 1 & 1 \end{pmatrix}$

2. Write a square matrix of order 2. Verify that the determinant is the product of its eigenvalues.

3. Determine whether the vectors $(4, -8), (-6, 12)$ are linearly independent or linearly dependent in $\mathbb{R}^2$.

4. Is the set of rational numbers $\mathbb{Q}$ a vector space over $\mathbb{R}$? Give reasons.

5. Find all the subspaces of the vector space $\mathbb{R}^2$.

6. What is the dimension of the vector space $\mathbb{C}^2$ over $\mathbb{R}$?

7. Determine whether the function $T : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $T(x, y) = (x + y, y)$ is a linear transformation

8. Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ is a linear transformation such that

$$T(1, 0, 0) = (2, 4, -1), T(0, 1, 0) = (1, 3, -2), T(0, 0, 1) = (0, -2, 2).$$

Compute $T(-2, 4, -1)$.

9. Determine whether $T(x, y) = (x + 2y, x - 2y)$ is invertible or not.

10. Suppose that U is a subspace of a vector space V. What is $U + U$?

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11. Give an example of an operator $T \in L(\mathbb{R}^2)$ such that T has no real eigenvalues.

12. Find the range and rank, for the differentiation transformation D (differentiation operator) on the space of polynomials of degree $\leq k$.

13. Define an inner product space. Define an inner product on $\mathbb{R}^3$.

14. State Cayley Hamilton Theorem.

15. Let V be a vector space and let $T \in L(V)$. Prove that the intersection of any collection of subspaces of V invariant under T is invariant under T.

SECTION B: Answer the following questions. Each carries five marks. (Ceiling 35 marks)

16. Use elementary row operations to reduce $\begin{pmatrix} 2 & 1 & -1 & 3 \\ 1 & -1 & 2 & 1 \\ -4 & 6 & -7 & 1 \\ 2 & 0 & 1 & 3 \end{pmatrix}$ to row-echelon form.

17. Find the eigenvalues and eigen vector of any one eigen value of:

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & 1 \\ 6 & -1 & 0 \\ -1 & -2 & -1 \end{pmatrix}$$

18. Suppose V is finite dimensional and U is a subspace of V. Then prove that there is a subspace W of V such that $V = U \oplus W$.

19. Prove that the real vector space consisting of all continuous real-valued functions on the interval $[0, 1]$ is infinite-dimensional.

20. Prove that two finite-dimensional vector spaces are isomorphic if and only if they have the same dimension.

21. Let $T \in \mathcal{L}(V)$. Suppose $\lambda_1, \ldots, \lambda_m$ are distinct eigen values of T and $v_1, \ldots, v_m$ are corresponding nonzero eigen vectors. Then prove that $(v_1, \ldots, v_m)$ is linearly independent.

22. Suppose $S, T \in \mathcal{L}(V)$. Prove that ST and TS have the same eigenvalues.

23. Give an example of a function $f : \mathbf{R}^2 \to \mathbf{R}$ such that

$$f(av) = af(v)$$

for all $a \in \mathbf{R}$ and all $v \in \mathbf{R}^2$ but f is not linear.

SECTION C: Answer any two questions. Each carries ten marks.

24. (a) Suppose V is finite dimensional. If $T \in \mathcal{L}(V)$, then prove that the following are equivalent:

    (i) T is invertible;

    (ii) T is injective;

    (iii) T is surjective.

    (b) Prove or give a counter example: if U is a subspace of V that is invariant under every operator on V, then $U = \{0\}$ or $U = V$.

25. (a) If u, v are orthogonal vectors in an inner product space V, then prove that

$$\|u + v\|^2 = \|u\|^2 + \|v\|^2$$

    (b) Suppose V is finite-dimensional and U is a subspace of V such that $\dim U = \dim V$. Prove that $U = V$.

26. (a) Find all solutions of the linear system:

$$\begin{aligned} x_1 + 2x_2 - x_3 &= 1 \\ 2x_1 + x_2 + 4x_3 &= 2 \\ 3x_1 + 3x_2 + 4x_3 &= 1 \end{aligned}$$

    (b) Prove that the span of two linearly independent vectors $(3, -1, 2)$, $(1, 2, -1)$ in $\mathbb{R}^3$ is a plane through the origin.

27. (a) Consider a real vector space P of polynomials defined on the interval $[a, b]$, and let $p_1, p_2 \in P$. Define

$$< p_1, p_2 >= \int_a^b p_1(x)p_2(x)dx$$

    Prove that $<\ \ >$ is an inner product on P.

    (b) Suppose $T \in \mathcal{L}(V)$. Suppose $S \in \mathcal{L}(V)$ is invertible. Prove that T and $S^{-1}TS$ have the same eigenvalues.

($2 \times 10 = 20$ Marks)