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D6BMT2202 **Reg. No.............................**

Name: ....

SIXTH SEMESTER B.Sc. DEGREE EXAMINATION, APRIL 2025
(Regular/Improvement/Supplementary)
MATHEMATICS
GMAT6B11T: COMPLEX ANALYSIS

Time: 2½ Hours **Maximum Marks: 80**

SECTION A: Answer the following questions. Each carries *two* marks.
(Ceiling 25 marks)

1. State the residue theorem.
2. Evaluate $\int_\gamma f(z)\,dz$ where $f(z) = \frac{z^2-9}{\cosh z}$ and $\gamma: |z| = 1$.
3. State the Casorati-Weierstrass Theorem.
4. Evaluate $\int_\gamma f(z)\,dz$ where $f(z) = z^2 + \frac{1}{z-4}$ and $\gamma: |z| = 1$.
5. Describe $e^{\log z}$.
6. Determine the length of the circumference of the circle $\{re^{it} : 0 \le t \le 2\pi\}$.
7. Define integral of a function $f(z)$ along a smooth curve γ.
8. State ML-inequality.
9. Define uniform convergence of a sequence of complex functions.
10. Expand $\sinh z$ and write hyperbolic sine function in terms of exponential function.
11. State Fundamental Theorem of Algebra.
12. Show that $|zw| = |z|\,|w|$.
13. Express $\frac{z+2}{z+1}$ in standard form where $z = x + iy$, with $x, y\ in\ \mathbb{R}$.
14. Let γ be a contour, let f be holomorphic in a domain containing $I(\gamma) \cup \gamma^*$, and let $a \in I(\gamma)$. What is the formula to evaluate $f^n(a)$.
15. Show that $\cos\frac{1}{z}$ has an essential singularity at 0.

SECTION B: Answer the following questions. Each carries *five* marks.
(Ceiling 35 marks)

16. Expand as Laurent's series valid for $1 < |z| < 3$ for the function $\frac{1}{(z-1)(z+3)}$.

(PTO)

17. Let γ^* be the top half of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, traversed in the positive (counter clockwise) direction. Determine $\int_\gamma \cos z\, dz$. (Apply Fundamental Theorem of Calculus).

18. Prove that, if $z, w \in \mathbb{C}$, then $|z+w|^2 + |z-w|^2 = 2(|z|^2 + |w|^2)$.

19. Let γ_1, γ_2 be contours, with γ_2 lying wholly inside γ_1, and suppose that f is holomorphic in a domain containing the region between γ_1 and γ_2. Prove that

$$\int_{\gamma_1} f(z)\, dz = \int_{\gamma_2} f(z)\, dz.$$

20. Let f be continuous on an open set D and $\int_\gamma f(z)\, dz = 0$ for every contour contained in D, prove that f is holomorphic in D.

21. Show that $cos(iz) = cosh\, z\ and\ sin(iz) = i\, sinh\, z$.

22. Evaluate $\int_\gamma 1/(z^4+1)\ dz$, where γ is the semicircle $[-R, R]\ U\ \{z : |z| = R, Imz > 0\}$, traversed in the positive direction, with $R > 1$.

23. For each $n \geq 1$, let f_n be a complex function with domain S, and suppose that there exist positive numbers M_n $(n \geq 1)$ such that $\|f_n\| \leq M_n$. If $\sum_{n=1}^{\infty} M_n$ is convergent, then prove that $\sum_{n=1}^{\infty} f_n$ is uniformly convergent in S.

SECTION C: Answer any *two* questions. Each carries *ten* marks.

24. Evaluate $\int_0^\infty \frac{x \sin mx}{a^2+x^2} dx,\ a > 0,\ m > 0$.

25. State and prove the necessary condition for a complex function to be differentiable at a point c.

26. Define a rectifiable curve. Show that the curve $C = \{(t, r(t)) : t \in [0,1]\}$ where $r(t) = \begin{cases} t\sin(1/t) & if\ t \neq 0 \\ 0 & if\ t = 0 \end{cases}$ is not rectifiable.

27. Let $c \in \mathbb{C}$ and suppose that the function f is holomorphic in some neighborhood $N(c, R)$ of c. Prove that, within $N(c, R)$, $f(z) = \sum_{n=0}^{\infty} a_n(z-a)^n$, where, for $n = 0,1,2, \dots, a_n = \frac{f^{(n)}(c)}{n!}$.

(2 x 10 = 20 Marks)