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D6BMT1802 (S3) **(PAGES 2)** **Reg. No..............................**

Name: ....

SIXTH SEMESTER B. Sc. DEGREE EXAMINATION, APRIL 2024

(Supplementary - 2018 Admission)

MATHEMATICS

AMAT6B10T: COMPLEX ANALYSIS

Time: 3 Hours. **Maximum Marks: 120**

PART A: Answer *all* the questions. Each carries *one* mark.

1. Write the function $f(z) = z^2 + 1$ in the form $f(z) = u(x, y) + iv(x, y)$.
2. When a function f of the complex variable z is analytic at a point z_0 ?
3. If e^z is real, then **Im**z, the imaginary part of z, is.....
4. Evaluate $\int_0^{\frac{\pi}{6}} e^{i2t}\, dt$.
5. Define a simply connected domain.
6. State Cauchy integral formula.
7. When an infinite sequence of complex numbers $z_1, z_2, z_3, \dots, z_n, \dots$ has a limit z?
8. State True or False: "Any function which is analytic at a point z_0 must have a Taylor series about z_0."
9. Define the radius of convergence of a power series.
10. The number of isolated singular points of the function $f(z) = \frac{z+1}{z^3(z^2+1)}$ is.....
11. Find the residue at $z = 0$ of the function $f(z) = \frac{z-\sin z}{z}$.
12. Let function f be analytic at a point z_0. When f is said to have a zero of order m at z_0?

(12 x 1 = 12 Marks)

PART B: Answer any *ten* questions. Each carries *four* marks.

13. Verify Cauchy-Riemann equations for the function $f(z) = z^2$.
14. Show that $u(x, y) = \sinh x \sin y$ is harmonic.
15. Show that $\exp(2 \pm 3\pi i) = -e^2$.
16. Show that if **Re** $z_1 > 0$ and **Re** $z_2 > 0$, then $\text{Log}(z_1 z_2) = \text{Log} z_1 + \text{Log} z_2$.
17. Evaluate $\int_C f(z)dz$ where $f(z) = \frac{z+2}{z}$ and C is the circle $z = 2e^{i\theta}, 0 \le \theta \le 2\pi$.
18. Using Cauchy-Goursat theorem, show that $\int_C \frac{1}{z^2+2z+2} dz = 0$ when the contour C is the circle $|z| = 1$, in either direction.
19. Find the value of the integral of $f(z) = \frac{1}{z^2+4}$ around the circle $|z - i| = 2$ in the positive sense.
20. Obtain the Maclaurin series representation $\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n, |z| < 1$.
21. State Laurent's theorem.
22. Find a representation of the function $f(z) = \frac{1}{1+z}$ in negative powers of z that is valid when $1 < |z| < \infty$.
23. Evaluate the integral of the function $f(z) = \frac{z+1}{z^2-2z}$ around the circle $|z| = 3$ in the positive sense.

24. Show that the function $f(z) = \frac{\sinh z}{z^4}$ has a pole of order 3 at $z_0 = 0$.

25. Show that residue of $f(z) = \frac{z - \sinh z}{z^2 \sinh z}$ at $z = \pi i$ is $\frac{i}{\pi}$.

26. If z_0 is a removable singular point of a function f, then show that f is analytic and bounded in some deleted neighborhood $0 < |z - z_0| < \varepsilon$ of z_0.

(10 x 4 = 40 Marks)

PART C: Answer any *six* questions. Each carries *seven* marks.

27. If a function $f(z)$ is continuous and nonzero at a point z_0, then prove that $f(z) \neq 0$ throughout some neighborhood of that point.

28. If a function $f(z) = u(x,y) + iv(x,y)$ is analytic in a domain D, then prove that its component functions u and v are harmonic in D.

29. Show that $\tan^{-1} z = \frac{i}{2}\log\frac{i+z}{i-z}$.

30. If a function f is analytic at a point, then prove that its derivatives of all orders exist at that point. Also prove that those derivatives are, moreover, all analytic there.

31. State and prove Cauchy's inequality.

32. Represent the function $f(z) = \frac{z+1}{z-1}$ by its Laurent series in the domain $1 < |z| < \infty$.

33. If a series $\sum_{n=0}^{\infty} a_n(z - z_0)^n$ converges to $f(z)$ at all points interior to some circle $|z - z_0| = R$, then prove that it is the Taylor series expansion for f in powers of $z - z_0$.

34. Describe the three types of isolated singular points.

35. Find the value of the integral $\int_C \frac{dz}{z^3(z+4)}$ taken counterclockwise around the circle $|z| = 2$.

(6 x 7 = 42 Marks)

PART D: Answer any *two* questions. Each carries *thirteen* marks.

36. (a) Derive Cauchy-Riemann equations in polar coordinates.

    (b) When α is a fixed real number, show that the function $f(z) = \sqrt[3]{r}e^{-\frac{i\theta}{3}}$, $r > 0$, $\alpha < \theta < \alpha + 2\pi$, has a derivative everywhere in its domain of definition. Also find $f'(z)$.

37. State and prove Taylor's theorem.

38. (a) Evaluate the improper integral $\int_0^{\infty} \frac{dx}{x^4+1}$.

    (b) Show that $\int_0^{2\pi} \frac{d\theta}{1+a\sin\theta} = \frac{2\pi}{\sqrt{1-a^2}}$, $-1 < a < 1$.

(2 x 13 = 26 Marks)