8

D6BMT2103 (PAGES 2) Reg.No:......................
Name: ....

SIXTH SEMESTER B.Sc. DEGREE EXAMINATION, APRIL 2024
(Regular/Improvement/Supplementary)
MATHEMATICS
GMAT6B12T - LINEAR ALGEBRA

Time: 2.5 Hours **Maximum Marks: 80**

SECTION A: Answer the following questions. Each carries *two* marks.
(Ceiling 25 Marks)

1. Reduce the following matrix in echelon form.
$$\begin{pmatrix} 1 & -2 & 3 & -1 \\ 2 & -1 & 2 & 2 \\ 3 & 1 & 2 & 3 \end{pmatrix}.$$
2. Give an example of a vector space of dimension 8.
3. Sate Cayley Hamilton Theorem.
4. Define the dimension of a vector space. Let V be the vector space of all polynomials of degree less than or equal 3 over the field $\mathbb{R}$. What is the dimension of V ?
5. Construct a vector space with exactly 9 elements.
6. Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be defined by $T(x, y, z) = (x, x)$. Find the range of T.
7. Show that the linear map $\mathbb{R}^2 \to \mathbb{R}^2$ given by $T(x, y) = (x + y, x - y)$ is bijective.
8. Find the eigenvalues of the linear map $T : \mathbb{R}^2 \to \mathbb{R}^2$ given by $T(x, y) = (y, x)$.
9. Let V be the class of all polynomials of degree greater than 3 over a field F. Is it a vector space ? Justify your answer.
10. What do you mean by matrix of a linear map ? Explain with an example.
11. Can you extend the set $\{(1, 1, 1), (1, 0, 2)\}$ to a basis of $\mathbb{R}^3$? Give reason.
12. Let V and W be vector spaces over a field F. Suppose T is a linear map from V to W. Prove that $T(0) = 0$.
13. Let V be a vector space over a field F and, let $T \in \mathcal{L}(V)$. Let U be a subspace of V. When do you say that U is invariant under T ? Is null space of T invariant under T ?
14. Let $V = C[0, 1]$. For $f, g \in V$, put
$$\langle f, g\rangle = \int_0^1 \overline{f(t)}g(t)dt.$$
Prove that $\langle , \rangle$ is an inner product.
15. Define an inner product space. Give example of an inner product on $\mathbb{R}^2$.

SECTION B: Answer the following questions. Each carries *five* marks.
(Ceiling 35 Marks)

16. Solve the following system of homogeneous equations.
$$3x + 2y + 7z = 0$$
$$4x - 3y - 2z = 0$$
$$5x + 9y + 23z = 0$$

(PTO)

17. Let $A = \begin{bmatrix} 1 & 1 & 2 \\ 9 & 2 & 0 \\ 5 & 0 & 3 \end{bmatrix}$. Find inverse of A using Cayley-Hamilton theorem.

18. Suppose $T \in \mathcal{L}(\mathbf{F}^2)$ is defined by T(w, z)=(-z, w)

    (a) Find the eigenvalues and eigenvectors of T if $\mathbf{F} = \mathbf{R}$.

    (b) Find the eigenvalues and eigenvectors of T if $\mathbf{F} = \mathbf{C}$.

19. Show that the function that takes $((x_1, x_2), (y_1, y_2)) \in \mathbf{R}^2 \times \mathbf{R}^2$ to $|x_1 y_1| + |x_2 y_2|$ is not an inner product on $\mathbf{R}^2$.

20. Suppose $T \in \mathcal{L}(V, W)$ is injective and $v_1, \ldots, v_n$ is linearly independent in V. Prove that $Tv_1, \ldots, Tv_n$ is linearly independent in W.

21. Define $T : \mathbf{R}^2 \to \mathbf{R}^2$ by

$$T(x, y) = (\sin x, y)$$

    Show that T is not a linear map.

22. Show that the subspaces of $\mathbf{R}$ are precisely $\{0\}$ and $\mathbf{R}$.

23. Let U be the subspace of $\mathbf{C}^5$ defined by $U = \{(z_1, z_2, z_3, z_4, z_5) \in \mathbf{C}^5 : 6z_1 = z_2 \text{ and } z_3 + 2z_4 + 3z_5 = 0\}$. Find a basis of U.

SECTION C: Answer any *two* questions. Each carries *ten* marks.

24. (a) Let $T \in \mathcal{L}(V)$. Suppose $\lambda_1, \ldots, \lambda_m$ are distinct eigenvalues of T and $v_1, \ldots, v_m$ are corresponding eigenvectors. Prove that $v_1, \ldots, v_m$ is linearly independent.

    (b) Prove that every finite-dimensional vector space has a basis

25. (a) Suppose V is finite-dimensional and U is a subspace of V. Then prove that there is a subspace W of V such that $U \oplus W = V$.

    (b) Suppose v_1, v_2, v_3, v_4 is a basis of V. Prove that

$$v_1 + v_2, v_2 + v_3, v_3 + v_4, v_4$$

    is also a basis of V.

26. (a) Prove that a invertible linear map has a unique inverse.

    (b) Suppose U and W are subspaces of a vector space V. Then prove that $U + W$ is a direct sum if and only if $U \cap W = \{0\}$.

27. (a) Prove that the union of two subspaces of V is a subspace of V if and only if one of the subspaces is contained in the other.

    (b) Suppose V is finite-dimensional, $T \in \mathcal{L}(V)$, and $\lambda \in F$. Then prove the following are equivalent:

        (i) λ is an eigenvalue of T;

        (ii) $T - \lambda I$ is not injective;

        (iii) $T - \lambda I$ is not surjective;

        (iv) $T - \lambda I$ is not invertible.

(2 x 10 = 20 Marks.)