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D6BMT2102 (PAGES 2) **Reg.No............................**

Name: ....

SIXTH SEMESTER B.Sc. DEGREE EXAMINATION, APRIL 2024

(Regular/Improvement/Supplementary)

MATHEMATICS

GMAT6B11T: COMPLEX ANALYSIS

Time: 2 ½ Hours **Maximum Marks: 80**

SECTION A: Answer the following questions. Each carries *two* marks.

(Ceiling 25 Marks)

1. Calculate the modulus and principal argument of $1 - i$.
2. Show that, for every pair c, d of non-zero complex numbers
$$\arg\left(\frac{c}{d}\right) \equiv \arg c - \arg d \ (mod\ 2\pi).$$
3. Express $\frac{3+7i}{2+5i}$ in standard form.
4. State Liouville's theorem.
5. When do you say that a complex function is differentiable at a point c. Write down the Cauchy-Riemann equations.
6. Let $S \subset \mathbb{C}$. Define closure, interior and boundary of the set S.
7. State Heine-Borel theorem.
8. Give the formula for finding the length of a curve $C = \{r(t): t \in [a, b]\}$.
9. Define $\tilde{\gamma}(t)$.
10. Evaluate $\int_\gamma 1/z^2\, dz$ where $\gamma: (x-2)^2 + \frac{1}{4}(y-5)^2 = 1$.
11. Evaluate $\int_\gamma f(z)\, dz$ where $f(z) = \frac{\sin z}{(z^2-25)(z^2+9)}$ and γ: $|z| = 1$.
12. State Morera's theorem.
13. Sketch the region defined by $|\leq|z|\leq|2, 0 \leq argz \leq \pi$
14. Define the term residue.
15. State the Casorati-Weierstrass Theorem.

SECTION B: Answer the following questions. Each carries *five* marks.

(Ceiling 35 Marks)

16. Show that $\mathbb{C}$ and ϕ are the only two subsets of $\mathbb{C}$ that are both open and closed.
17. Let S be a closed bounded set, and let f, with domain containing S, be continuous and nonzero throughout S. Prove that $\inf\{|f(z)| : x \in S\} > 0$.
18. Let $f(z) = \frac{z^3-4z+1}{(z^2+5)(z^3-3)}$ and $\gamma(t) = Re^{it}, 0 \leq t \leq \pi$. Show that
$\left|\int_\gamma f(z)\, dz\right| \leq \frac{\pi R(R^3+4R+1)}{(R^2+5)(R^3-3)}$.

(PTO)

19. Let $\gamma_1, \gamma_2 : [a,b] \to \mathbb{C}$ be piecewise smooth curves such that $\gamma_1(a) = \gamma_2(a)$, $\gamma_1(b) = \gamma_2(b)$, $\gamma_1(t_1) \neq \gamma_2(t_2)$ $(t_1, t_2 \in (a,b))$. If f is holomorphic throughout an open set containing γ_1^*, γ_2^* and the region between, prove that $\int_{\gamma_1} f(z)\,dz = \int_{\gamma_2} f(z)\,dz$.
20. Find the Maclaurin series expansion of $\log(1+z)$.
21. Show that the coefficient of z^{-1} in the Laurent series of $e^{\frac{1}{z}} e^{2z}$ is $\sum_{n=0}^{\infty} \frac{2^n}{n!(n+1)!}$
22. Determine $\int_\gamma 1/(z^2+1)^2\, dz$, where γ is the semicircle $[-R,R]\ U\ \{z : |z| = R, Imz > 0\}$, traversed in the positive direction, with $R > 1$.
23. Evaluate $\int_{\kappa(0,2)} \frac{\sin \pi z}{(2z+1)^3}\, dz$.

SECTION C: Answer any *two* questions. Each carries *ten* marks.

24. Show that a power series $\sum_{n=0}^{\infty} c_n (z-a)^n$ satisfies exactly one of the following three conditions:

    (i) the series converges for all z;

    (ii) the series converges only for $z = a$;

    (iii) there exists a positive real number R such that the series converges for all z such that $|z - a| < R$ and diverges for all z such that $|z - a| > R$.
25. Let $\gamma: [a,b] \to \mathbb{C}$ be piecewise smooth. Let F be a complex function defined on an open set containing γ^*, and suppose that $F'(z)$ exists and is continuous at each point of γ^*. Prove that $\int_\gamma F'(z)\,dz = F(\gamma(b)) - F(\gamma(a))$.
26. Let γ be a contour, let f be holomorphic in a domain containing $I(\gamma) \cup \gamma^*$, and let $a \in I(\gamma)$. Prove that $f'(a) = \frac{1}{2\pi i}\int_\gamma \frac{f(z)}{(z-a)^2}\, dz$.
27. Evaluate $\int_0^\pi \frac{1}{a^2+\cos^2\theta}\, d\theta, a > 1$.

(2 x 10 = 20 Marks)