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D6BMT2101 (PAGES 2) Reg.No............................

Name: ....

SIXTH SEMESTER B.Sc. DEGREE EXAMINATION, APRIL 2024
(Regular/Improvement/Supplementary)

MATHEMATICS
GMAT6B10T: ADVANCED REAL ANALYSIS

Time: 2 ½ Hours **Maximum Marks: 80**

SECTION A: Answer the following questions. Each carries *two* marks.
(Ceiling 25 Marks)

1. Let $I = [a,b]$ and let $f: I \to \mathbb{R}$ be continuous on I. If $f(a) < 0 < f(b)$, or if $f(a) > 0 > f(b)$, then prove that there exists a number $c \in (a,b)$ such that $f(c) = 0$
2. If $f(x) := x^2$on $A := [0,b]$, where $b > 0$, then show that f is uniformly continuous.
3. Define Riemann integral of a function.
4. If $\int_a^b f = 0$, can you conclude that $f(x) = 0 \forall x \in [a,b]$? Explain.
5. Write the function whose antiderivative is x^2
6. Show by an example that product of strictly increasing functions need not be an increasing function.
7. Find $\lim\left(\frac{x}{x+n}\right)$, for all $x \in \mathbb{R}, x \geq 0$.
8. Is pointwise limit of continuous functions is continuous?
9. Give an example of a series of functions which converges.
10. Define improper integral.
11. Write Cauchy's root test, for absolute convergence of a series.
12. What is the value of $\beta(3,4)$?
13. State substitution theorem for Riemann integrals .
14. If $I := [0,4]$, write the norm of the partition $P := (0, 2, 3, 4)$.
15. Give an example of Lipschitz function.

SECTION B: Answer the following questions. Each carries *five* marks.
(Ceiling 35 Marks)

16. Let I be a closed bounded interval and let $f : I \to \mathbb{R}$ be continuous on I. If $\varepsilon > 0$, then prove that there exists a step function $S_\varepsilon: I \to \mathbb{R}$ such that $|f(x) - s_\varepsilon(x)| < \varepsilon$ for all $x \in I$.
17. Let $F(x) = 1$ for $x = \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}$, and $F(x) = 0$ elsewhere on $[0,1]$. Then show that $F \in \mathcal{R}[0,1]$ and that $\int_0^1 F = 0$.
18. If $f \in \mathcal{R}[a,b]$, and define $F(z); = \int_a^z f$ for $z \in [a,b]$. If $|f(x)| \leq M$ for all $x \in [a,b]$, then show that $|F(z) - F(w)| \leq M|z - w|$ for all $z, w \in [a,b]$.
19. Let $I \subseteq \mathbb{R}$ be an interval and let $f: I \to \mathbb{R}$ be monotone on I. Then prove that the set of points $D \subseteq I$ at which f is discontinuous is a countable set.

(PTO)

20. Prove that a sequence (f_n) of functions on $A \subseteq \mathbb{R}$ to R does not converge uniformly on $A_0 \subseteq A$ to a function $f: A_0 \to \mathbb{R}$ on A_0 if and only if for each $\varepsilon_0 > 0$ there is a subsequence (f_{n_k}) of (f_n) and a sequence (x_k) in A_0 such that $|f_n(x_k) - f(x_k)| \geq \varepsilon_0$ for all $k \in \mathbb{N}$.
21. If f_n is continuous on $D \subseteq \mathbb{R}$ to $\mathbb{R}$ for each $n \in \mathbb{N}$ and if $\sum f_n$ converges to f uniformly on D, then show that f is continuous on D.
22. Find the value of $\int_{-\infty}^{\infty} \frac{2x}{1+x^2} dx$ if it exists.
23. Evaluate the integral $\int_0^{\infty} x^4 e^{-2x} dx$ using Gamma function.

SECTION C: Answer any *two* questions. Each carries *ten* marks.

24. State and prove Maximum-Minimum theorem.
25. If $f \in \mathcal{R}[a, b]$ and if $a = c_0 < c_1 < \cdots < c_m = b$, then show that the restrictions of f to each of the subintervals $[c_{i-1}, c_i]$ are Riemann integrable

$$\int_a^b f = \sum_{i=1}^{m} \int_{c_{i-1}}^{c_i} f$$

26. Let (f_n) be a sequence of continuous functions on a set $A \subseteq \mathbb{R}$ and suppose that (f_n) converges uniformly on A to a function $f: A \to \mathbb{R}$. Then prove that f is continuous on A.
27. Show that $\int_{\pi}^{\infty} \frac{\sin x}{x} dx$ converges conditionally.

(2 x 10 = 20 Marks)