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Reg. No....................................

D6BMT2003 (PAGES 3) Name.......................................

SIXTH SEMESTER B. Sc DEGREE EXAMINATION, APRIL 2023

(Regular/Improvement/Supplementary)

MATHEMATICS

GMAT6B12T - LINEAR ALGEBRA

Time: 2 ½ Hours **Maximum Marks : 80**

SECTION A : Answer the following questions. Each carries *two* marks.

(Ceiling 25 Marks)

1. Find the rank of the matrix $\begin{bmatrix} 1 & 1+i & -i \\ 0 & i & 1+2i \\ 1 & 1+2i & 1+i \end{bmatrix}$
2. Define homogeneous and non-homogeneous system of linear equations with examples.
3. Find the characteristic equation of $A = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}$
4. Show that $\dfrac{-1+\sqrt{3}i}{2}$ is a cube root of 1.
5. Prove that every element in a vector space has a unique additive inverse.
6. Define infinite dimensional vector space and give two examples.
7. True or False : A spanning list in a vector space may not be a basis. Give reason.
8. Let $T \in \mathcal{L}(V, W)$ and let null $T = \{0\}$. Prove that T is injective.
9. Let $T \in \mathcal{L}(\mathcal{P}(R), \mathcal{P}(R))$ is the linear map defined by $(Tp)(x) = x^2 p(x)$.
   Find the range of T.
10. Define innerproduct space on a vector space.
11. Give an example of a function $f : \mathbb{R}^2 \to \mathbb{R}$ such that $f(av) = af(v)$ for all $a \in \mathbb{R}$ and all $v \in \mathbb{R}^2$, but f is not linear.

(PTO)

12. Prove or give a counter example : if U is subspace of V that is invariant under every operator on V, then $U = \{0\}$ or $U = V$.

13. Define $T \in \mathcal{L}(F^2)$ by $T(w, z) = (z, w)$. Find all eigenvalues and eigenvectors of T.

14. True or False : Multiplication of linear maps is not commutative. Give reason.

15. Prove that an innerproduct space satisfies the conjugate homogeneity in the second slot.

SECTION B: Answer the following questions. Each carries *five* marks.

(Ceiling 35 Marks)

16. Reduce the matrix $A = \begin{bmatrix} 0 & 2 & 3 & 4 \\ 2 & 3 & 5 & 4 \\ 4 & 8 & 13 & 12 \end{bmatrix}$ to normal form.

17. Find all solutions of $\begin{cases} x_1 + x_2 + x_3 + x_4 = 0 \\ x_1 + x_2 + x_3 - x_4 = 4 \\ x_1 + x_2 - x_3 + x_4 = -4 \\ x_1 - x_2 + x_3 + x_4 = 2 \end{cases}$

18. Prove that the union of two subspaces of V is a subspace of V if and only if one of the subspaces is contained in the other.

19. Prove that if $\{v_1, v_2, \ldots, v_n\}$ spans V, then so does the list $\{v_1 - v_2, v_2 - v_3, \ldots, v_{n-1} - v_n, v_n\}$ obtained by subtracting from each vector (except the last one) the following vector.

20. Suppose $T \in \mathcal{L}(V, W)$ and $(v_1, \ldots, v_n)$ is a basis of V and $(w_1, \ldots, w_m)$ is a basis of W. Then prove that $\mathcal{M}(Tv) = \mathcal{M}(T)\mathcal{M}(v)$ for every $v \in V$.

21. Suppose V is finite dimensional. If $T \in \mathcal{L}(V)$, then prove that the following are equivalent :

    (a) T is invertible

    (b) T is injective

    (c) T is surjective

22. Let $T \in \mathcal{L}(V)$. Suppose $\lambda_1, \ldots, \lambda_m$ are distinct eigenvalues of T and $v_1, \ldots, v_m$ are corresponding nonzero eigenvectors. Then show that $(v_1, \ldots, v_m)$ is linearly independent.

23. Give an example of an operator whose matrix with respect to some basis contains only 0's on the diagonal, but the operator is invertible.

SECTION C: Answer any *two* questions. Each carries 10 marks.

24. (a) Find the characteristic roots and the associated invariant vectors of $A = \begin{bmatrix} 1 & -4 & -1 & -4 \\ 2 & 0 & 5 & -4 \\ -1 & 1 & -2 & 3 \\ -1 & 4 & -1 & 6 \end{bmatrix}$

(b) State Cayley Hamilton Theorem. Use the theorem to compute A^{-1} for the matrix

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 3 & 1 & 1 \\ 2 & 3 & 1 \end{bmatrix}$$

25. (a) Prove that the real vector space consisting of all continuous real valued functions on the interval $[0, 1]$ is infinite dimensional.

(b) Let U be the subspace of $\mathbb{R}^5$ defined by

$U = \{(x_1, x_2, x_3, x_4, x_5) \in \mathbb{R}^5 : x_1 = 3x_2 \text{ and } x_3 = 7x_4\}$. Find a basis of U .

(c) Suppose that V is finite dimensional and U is a subspace of V such that dim U = dim V. Prove that $U = V$

26. (a) Suppose that V is finite dimensional and $S, T \in \mathcal{L}(V)$. Prove that ST is invertible if and only if both S and T are invertible.

(b) Suppose that V is finite dimensional and $T \in \mathcal{L}(V)$. Prove that T is a scalar multiple of the identity if and only if $ST = TS$ for every $S \in \mathcal{L}(V)$.

27. (a) Suppose $P \in \mathcal{L}(V)$ and $P^2 = P$. Prove that V = null $P \oplus$ range P.

(b) Let $M_{m \times n}(C)$ be the complex vector space of $m \times n$ complex matrices.

Define $\langle A, B \rangle = \text{tr}\,(B^* A)$. Prove that $\langle , \rangle$ is an innerproduct.

(2 x 10 = 20 Marks)