D5BMT2301 Reg.No.............................

Name: ....

FIFTH SEMESTER UG DEGREE EXAMINATION NOVEMBER 2025
(Regular/Improvement/Supplementary)
B.Sc. MATHEMATICS
GMAT5B05T - ABSTRACT ALGEBRA

Time: 2.5 Hours **Maximum Marks: 80**

SECTION A: Answer the following questions. Each carries two marks (Ceiling 25 marks)

1. Give example of a non-abelian group.
2. Define an equivalence relation and give one example.
3. Construct the addition and multiplication tables for the following set $\mathbf{Z}_4$.
4. If G is a group and $a, b \in G$, then prove that each of the equations $ax = b$ and $xa = b$ has a unique solution.
5. Define a cyclic group. Give an example of a cyclic group.
6. Give an example of a group with exactly 4 subgroups.
7. Prove that the additive group $(\mathbf{Q}, +)$ is not cyclic.
8. Define Klein four-group. Prove that Klein four-group is not cyclic.
9. How many group homomorphisms are there from $\mathbf{Z}_6$ to $\mathbf{Z}$.
10. Find all cyclic subgroups of $\mathbf{Z}_6 \times \mathbf{Z}_3$.
11. Define the n^{th} dihedral group $\mathbf{D}_n$. What are the subgroups of D_4 ?
12. Define the group S_3. What are the normal subgroups of S_3?
13. Define a commutative ring. Give an example of a commutative ring.
14. State first isomorphism theorem.
15. Define the center of a group G. Is it a subgroup of G?. What is the center of the Klein four group.

(PTO)

SECTION B: Answer the following questions. Each carries five marks (Ceiling 35 marks)

16. If $(a, n) = 1$, then prove that $a^{\varphi(n)} \equiv 1(\text{mod} n)$.

17. Define the permutation group S_n. For $\alpha, \beta \in S_3$, let $\alpha \sim \beta$ if there exists $\sigma \in S_3$ such that $\sigma\alpha\sigma^{-1} = \beta$. Show that $\sim$ is an equivalence relation on S_3.

18. State and prove Lagrange's theorem.

19. Prove that any group of prime order is cyclic.

20. Prove that every subgroup of a cyclic group is cyclic. Is $(\mathbf{R}, +)$ cyclic?

21. Prove that every group is isomorphic to a permutation group.

22. Prove that $\text{Aut}(\mathbf{Z}) \cong \mathbf{Z}_2$ and $\text{Inn}(\mathbf{Z}) = \{1\}$.

23. Define Kernel of a homomorphism $\varphi : G \to G'$. Prove that Kernel of φ is a normal subgroup of G'.

SECTION C: Answer any two questions. Each carries ten marks.

24. (a) Prove that set of matrices, with entries from $\mathbf{Z}_2$, forms a field under the operations of matrix addition and multiplication.

    (b) If m, n are positive integers such that $\gcd(m, n) = 1$, then prove that $\mathbf{Z}_{mn}$ is isomorphic to $\mathbf{Z}_m \times \mathbf{Z}_n$.

25. (a) Let G be a group, and let H and K be subgroups of G. If $h^{-1}kh \in K$ for all $h \in H$ and $k \in K$, then prove that HK is a subgroup of G.

    (b) Prove that any permutation in S_n, where $n \geq 2$, can be written as a product of transpositions.

26. (a) Prove that the intersection of two normal subgroups is a normal subgroup.

    (b) Let $\phi : G_1 \to G_2$ be an isomorphism of groups.

    (a) If a has order n in G_1, then $\phi(a)$ has order n in G_2.

    (b) If G_1 is abelian, then so is G_2.

    (c) If G_1 is cyclic, then so is G_2.

27. (a) Prove that any finite integral domain is a field.

    (b) If N is a normal subgroup of G, then prove that the set of left cosets of N forms a group under the coset multiplication given by $aNbN = abN$ for all $a, b \in G$.

(2 X 10 = 20 Marks)