D5BMT2201 **Reg. No.............................**

Name: ....

FIFTH SEMESTER B.Sc. DEGREE EXAMINATION, NOVEMBER 2024

(Regular/Improvement/Supplementary)

MATHEMATICS

GMAT5B05T: ABSTRACT ALGEBRA

Time: 2 ½ Hours **Maximum Marks: 80**

SECTION A: Answer the following questions. Each carries *two* marks.
(Ceiling 25 marks)

1. Make addition and multiplication tables of $\mathbf{Z}_4$
2. Let S be the set of all ordered pairs (m,n) of positive integers. For $(a_1,a_2)\in S$ and $(b_1,b_2)\in S$, define $(a_1,a_2) \sim (b_1,b_2)$ if $a_1+b_2=a_2+b_1$. Show that ~ is an equivalence relation.
3. Let $\sigma=\begin{pmatrix}1&2&3&4\\4&3&1&2\end{pmatrix}$ and $\tau=\begin{pmatrix}1&2&3&4\\2&3&4&1\end{pmatrix}$. Verify that $(\sigma\tau)^{-1}=\tau^{-1}\sigma^{-1}$.
4. Write $\begin{pmatrix}1&2&3&4&5&6&7&8&9&10\\6&2&5&9&3&10&8&4&7&1\end{pmatrix}$ as a product of disjoint cycles.
5. Let $\sigma=(2\ \ 5\ \ 6\ \ 8\ \ 12\ \ 4\ \ 7\ \ 9\ \ 3\ \ 11)$ be a cycle in S_{12}. Is σ an even transposition? Why or why not?
6. Define $*$on $\mathbf{Z}$ by $a*b=\max\{a,b\}$. Determine whether or not $*$gives a group structure on $\mathbf{Z}$. If it is not a group, say which axiom fail to hold.
7. Prove that S_3 is not a cyclic group.
8. Give an example for a group of order 6 which is not cyclic.
9. Give the subgroup diagram of $\mathbb{Z}_{28}$.
10. Let $\langle\mathbb{R}^{\times}, .\rangle$ be set of nonzero real numbers under multiplication and $\langle\mathbb{R}^{+}, .\rangle$ be set of positive real numbers under multiplication. Define $\phi:\ \mathbb{R}^{\times}\to\mathbb{R}^{+}$ by $\phi(x)=|x|$. Prove that ϕ is a homomorphism.
11. Let $G=\mathbf{Z}_3\times\mathbf{Z}_6$ and let $H=\langle(1,2)\rangle$. List all cosets of H.
12. Is the group $\mathbf{Z}_p$ simple? Why or why not?
13. Find the multiplicative inverse of $(1-\sqrt{2})$ in the field $\mathbb{Q}(\sqrt{2})$.
14. Find $Aut(\mathbf{Z})$.
15. Is $A=\{m+n\sqrt{2}\,|\,m,n\in\mathbf{Z}\text{ and m is odd}\}$ subring of the field $\mathbf{R}$ of real numbers? Why or why not?

(PTO)

SECTION B: Answer the following questions. Each carries *five* marks
(Ceiling 35 marks)

16. Let S be any set and let σ *and* τ be disjoint cycles *in* $\text{Sym}(S)$. Prove that $\sigma\tau = \tau\sigma$.

17. Let G be a group, and suppose that a and b are any elements of G. Prove that $(ab)^2 = a^2b^2$ if and only if $ab = ba$.

18. Show that $H = \left\{ \begin{bmatrix} a & 0 \\ c & d \end{bmatrix} \middle| ad \neq 0 \right\}$ is a subgroup of $GL_2(\mathbb{R})$.

19. Find the cyclic subgroup generated by $\begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$ in $GL_2(\mathbf{Z}_3)$.

20. Let G be a cyclic group and let H be a nonempty subgroup of G. Prove that H is cyclic.

21. Let G_1 and G_2 be groups and let $\phi: G_1 \to G_2$ be an onto homomorphism. Prove that there exists an isomorphism $\bar{\phi}: G_1/\ker(\phi) \to G_2$.

22. Let $\mathbb{F}$ be a field, $f(x) \in \mathbb{F}[x]$ be a nonzero polynomial, and let $c \in \mathbb{F}$. Prove that there exists a polynomial $q(x) \in \mathbb{F}[x]$ such that $f(x) = q(x)(x - c) + f(c)$.

23. Let R be a commutative ring and let $R^\times$ be the set of all units of R. Prove that $R^\times$ is an abelian group under the multiplication of R.

SECTION C: Answer any *two* questions. Each carries *ten* marks.

24. (i) Find the multiplicative inverse of [91] in $\mathbb{Z}_{2565}$.
   (ii) Let G be a group. Let a be an element of G with $o(a) = m$ and let $k \in \mathbb{Z}$. Prove that $a^k = e$ if and only if $m|k$.
   (iii) Find all normal subgroups of S_3.

25. (i) Let G be a group and let H be a subgroup of G. For $a, b \in G$ define $a \sim b$ if $ab^{-1} \in H$. Prove that $\sim$ is an equivalence relation.

   (ii) Let G be a finite group and let H be a subgroup of G. Prove that the order of H is a divisor of the order of G.

   (iii) Let $G = \mathbb{Z}_{21}^\times$. If $H = \{[1], [8]\}$and $K = \{[1], [4], [10], [13], [16], [19]\}$, then find HK and KH. Is $HK = KH$?

26. (i) Define permutation group.

   (ii) Prove that every group is isomorphic to a permutation group.

27. (i) Compute the factor group $\dfrac{(\mathbf{Z}_6 \times \mathbf{Z}_4)}{\langle (2,2) \rangle}$.

   (ii) Let G be a group and let H be a normal subgroup of G. Prove that abH is the set theoretic product $(aH)(bH)$, for all $a, b \in G$.

   (iii) Let $\mathbb{F}$ be a field and let R be a subring of $\mathbb{F}$. Prove that R is an integral domain.

(2 × 10 = 20 Marks)