D5BMC2205 **Reg. No.............................**

Name: ....

FIFTH SEMESTER B.Sc. DEGREE EXAMINATION, NOVEMBER 2024
COMPUTER SCIENCE AND MATHEMATICS (DOUBLE MAIN)
GDMA5B09T: ALGEBRA

Time: 2 ½ Hours **Maximum Marks: 80**

SECTION A: Answer the following questions. Each carries *two* marks.
(Ceiling 25 marks)

1. Find the subgroup of $\mathbb{Z}_4$ generated by 3.
2. Show that for any integer $k \neq 0$, $\gcd(ka, kb) = |k| \gcd(a, b)$.
3. Find $\tau(12)$ and $\sigma(12)$.
4. Show that $(n\mathbb{Z}, +)$ is a group.
5. Give an example of an abelian group that is not cyclic.
6. Determine whether the binary operation $*$ defined on $\mathbb{Z}$ by letting $a * b = ab$ gives a group structure.
7. Define even permutation. Give one example.
8. What is the order of the groups S_n and alternating group A_n.
9. Find the gcd of 42 and 72.
10. For any integers a and b, show that if $a|b$ and $b \neq 0$ then $|a| \leq |b|$.
11. Define binary operation and give one example.
12. If $a \equiv b \ mod(n)$, show that $a^k \equiv b^k \ mod(n)$, for any positive integer k.
13. Show that if a and b are two integers, not both zero. Then a and b are relatively prime if and only if there exist integers x and y such that $1 = ax + by$.
14. Prove that if $a|c$ and $b|c$, with $\gcd(a, b) = 1$, then $ab|c$.
15. Prove that if p is a prime and $p|ab$, then $p|a$ or $p|b$.

SECTION B: Answer the following questions. Each carries *five* marks.
(Ceiling 35 marks)

16. State division algorithm theorem. Show that the expression $\frac{a(a^2+2)}{3}$ is an integer for all $a \geq 1$.
17. Find the remainder obtained upon dividing the sum $1! + 2! + 3! + \cdots + 99! + 100!$ by 12.
18. Solve the system of congruences:

    $7x + 3y \equiv 10(mod\ 16),\ \ 2x + 5y \equiv 9(mod\ 16),\ \ .$

(PTO)

19. Prove that if p is a prime, then $(p-1)! \equiv -1 \, mod(p)$.

20. Let G be any group. Show that $H = \{x \in G |\, xg = gx \; for \; all \; g \in G\}$ is a subgroup of G.

21. If G is a finite group of order n, then show that G is isomorphic to $(\mathbb{Z}_n, +_n)$.

22. Show by any example that every proper subgroup of a nonabelian group may be abelian.

23. Find all the subgroups of $\mathbb{Z}_{18}$ and draw a subgroup diagram.

SECTION C: Answer any *two* questions. Each carries *ten* marks.

24. State and prove the Fundamental Theorem of Arithmetic.

25. Let G be a group with identity e. Then

    a) Show that if $x * x = e$ for all $x \in G$ then G is abelian.

    b) If $(a * b)^2 = a^2 * b^2$ for all $a, b \in G$, then G is abelian.

26. Show that every group is isomorphic to a group of permutations.

27. State and prove Chinese Reminder Theorem.

(2 × 10 = 20 Marks)