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D5BMC2204 **Reg. No.............................**

Name: ....

FIFTH SEMESTER B.Sc. DEGREE EXAMINATION, NOVEMBER 2024
COMPUTER SCIENCE AND MATHEMATICS (DOUBLE MAIN)
GDMA5B08T: REAL ANALYSIS

Time: 2 ½ Hours **Maximum Marks: 80**

SECTION A: Answer the following questions. Each carries *two* marks.
(Ceiling 25 marks)

1. Define uncountable set and give one example.
2. If $S = \left\{\frac{1}{n} : n \in \mathbb{N}\right\}$, then prove that $\inf S = 0$.
3. State the nested intervals property.
4. State Characterization of Closed Sets.
5. Show that $lim\left(\frac{1}{n}\right) = 0$ by using the definition of limit.
6. Show that the sequence $(\frac{1}{n})$ is a Cauchy sequence.
7. Prove that if a sequence $X = (x_n)$ of real numbers converges to a real number x, then any subsequence $X' = (x_{nk})$ of X also converges to x.
8. State Monotone Subsequence Theorem and the Bolzano -Weierstrass Theorem.
9. Give examples for a convergent sequence and a divergent sequence.
10. Check the convergence of the series $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$.
11. Calculate the value of $\sum_{n=1}^{\infty} \left(\frac{1}{3}\right)^{2n}$.
12. Define open set and closed set in $\mathbb{R}$.
13. Show that the entire set $\mathbb{R} = (-\infty, \infty)$ is open.
14. Define a sequence in $\mathbb{R}$ and give one example.
15. Define open cover of a subset of $\mathbb{R}$.

SECTION B: Answer the following questions. Each carries *five* marks.
(Ceiling 35 Marks)

16. Check the convergence of the geometrical series.
17. Show that every contractive sequence is a Cauchy sequence, and therefore is convergent.

(PTO)

18. Show that $lim\left(\frac{1}{n^2+1}\right) = 0$ by using the definition of limit.

19. Let $X = (x_n : n \in \mathbb{N})$ be a sequence of real numbers and let $m \in \mathbb{N}$. Then prove that the m-tail $X_m = (x_{m+n} : n \in \mathbb{N})$ of X converges if and only if X converges.

    In this case, $lim\, X_m = lim\, X$.

20. Let S be a nonempty subset of $\mathbb{R}$ that is bounded above, and let a be any number in $\mathbb{R}$. Define the set $a + S = \{a + s : s \in S\}$.
    Prove that $\sup(a + S) = a + \sup S$.

21. Use Mathematical Induction to prove that if the set S has n elements, then $P(S)$ has 2^n elements.

22. (a) Define Cauchy sequence.

    (b) Show that the sequence $(1 + (-1)^n)$ is not a Cauchy sequence.

23. Illustrate with an example that the intersection of infinitely many open sets in $\mathbb{R}$ need not be open.

SECTION C: Answer any *two* questions. Each carries *ten* marks.

24. State and prove Characterization of Closed Sets.

25. (a) Show that if $c > 0$, then $\lim\left(c^{\frac{1}{n}}\right) = 1$.

    (b) Show that $\lim\left(n^{1/n}\right) = 1$.

26. (a) Let $X = (x_n)$ be a sequence of real numbers, then prove that the following are equivalent:

    (i) The sequence $X = (x_n)$ does not converge to $x \in \mathbb{R}$.

    (ii) There exists an $\varepsilon_0 > 0$ such that for any $k \in \mathbb{N}$, there exists $n_k \in \mathbb{N}$ such that $n_k \geq k$ and $|x_{nk} - x| \geq \varepsilon_0$.

    (iii) There exists an $\varepsilon_0 > 0$ and a subsequence $X' = (x_{nk})$ of X such that

    $|x_{nk} - x| \geq \varepsilon_0$ for all $k \in \mathbb{N}$.

    (b) State the Divergence Criterion of a sequence.

27. Prove that the following statements are equivalent:
    (a) S is a countable set.
    (b) There exists a surjection of $\mathbb{N}$ onto S.
    (c) There exists an injection of S into $\mathbb{N}$.

(2 X 10 = 20 Marks)