D5BHM2201 (PAGES: 2) **Reg. No.............................**

Name: ....

FIFTH SEMESTER B.Sc. DEGREE EXAMINATION, NOVEMBER 2024

HONOURS IN MATHEMATICS

GMAH5B20T: ALGEBRA I

Time: 3 Hours **Maximum Marks: 80**

Part A. Answer *all* the questions. Each question carries *one* mark.

Choose the correct answer.

1. The identity element in the group $< \mathbb{Z} , + >$ is __________

   A) 2 B) 0 C) 1 D) -1

2. Which of the following is not a requirement for a set G with a binary operation $*$ to be considered a group?

   A) Existence of identity element. B) Commutative under $*$

   C) Closed under $*$ D) Existence of inverse element.

3. Which of the following is a generator of the cyclic group $\mathbb{Z}_6$.

   A) 1 B) 2 C) 3 D) All of the above

4. The order of the alternating group A_5 is __________.

   A) 60 B) 120 C) 24 D) 720

5. If G is a group of order 12, which of the following cannot be the order of a subgroup of G?

   A) 1 B) 2 C) 3 D) 7

Fill in the Blanks.

6. A binary operation $*$ on a set S is a function that assigns to each ordered pair (a, b) of elements in S a unique element in _______.

7. A permutation is called even if it can be expressed as a product of an _______ number of transpositions.

8. The group $\mathbb{Z}_n$ is a cyclic group of order _______.

9. If H is a subgroup of G, then the number of left cosets of H in G is called the _______ of H in G.

10. A group is simple if it is nontrivial and __________.

(10 × 1 =10 Marks)

Part B. Answer any *eight* questions. Each question carries *two* marks.

11. Give an example of a binary operation on $\mathbb{Z}$ and prove that it is a binary operation.

12. Find all abelian groups of order 10 up to isomorphism.

13. Show that the set of integers under addition forms a group.

(PTO)

14. Prove that the inverse element is unique in a group.
15. Find the order of the factor group $\mathbb{Z}_6 / < 3 >$.
16. Define the kernel of a homomorphism.
17. Prove that the order of an element of a finite group divides the order of a group.
18. Prove that the addition operation is associative on the set of real numbers.
19. Define the orbit of an element under a permutation.
20. Find the inverse of the permutation (1 3 2) in S_3.

(8 × 2 = 16 Marks)

Part C. Answer any *six* questions. Each question carries *four* marks.

21. Let X be a G-set. For each g ∈ G, the function $\sigma_g : X \longrightarrow X$ defined by $\sigma_g(x) = gx$ for x ∈ X is a permutation of X.
22. Prove that M is a maximal normal subgroup of G if and only if G/M is simple.
23. Prove that every cyclic group is abelian. Is the converse true? Justify your answer.
24. Identify a non-trivial subgroup of the group $< \mathbb{Z}, + >$ and prove that it satisfies the subgroup conditions.
25. Consider the binary operation $*$ defined on the set of real numbers by $a * b = ab + 1$. Find the identity element in R under $*$.
26. Check whether $\mathbb{Z}_3 \times \mathbb{Z}_3$ is cyclic. Justify your answer.
27. Show that $< \mathbb{Z} , + >$ is isomorphic to $< 2\mathbb{Z} , + >$.
28. Define an abelian group and give an example.

(6 × 4 = 24 Marks)

Part D. Answer any *two* questions. Each carries *fifteen* marks.

29. State and prove Cayley's theorem.
30. a) List the elements of $\mathbb{Z}_2 \times \mathbb{Z}_4$. Find the order of each of the elements.

    b) Prove that if m divides the finite abelian group G, then G has a subgroup of order m.
31. State Lagrange's theorem and prove the falsity of the converse of Lagrange's theorem.

(2 × 15 = 30 Marks)