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D5BMT2001 (2 Pages) Name..........................

Reg.No..........................

FIFTH SEMESTER UG DEGREE EXAMINATION, NOVEMBER 2022

(Regular/Improvement/Supplementary)

MATHEMATICS

GMAT5B05T - ABSTRACT ALGEBRA

Time: $2\frac{1}{2}$ *Hours* Maximum: 80 Marks

SECTION A: Answer the following questions. Each carries 2 marks (Ceiling 25 Marks)

1. Find all divisors of zero in $\mathbb{Z}_{14}$.
2. Compute $(1,4,2,5)(2,6,3)$.
3. Show by an example that the product of two cycles need not be a cycle.
4. If G is a nonempty set with an associative binary operation in which the equations $ax = b$ and $xa = b$ have a unique solutions for all $a, b \in G$, then prove that G is a group.
5. Let G be an abelian group. Show that the set of all elements of G of finite order forms a subgroup of G.
6. If F be a field, then show that $GL_n(F)$ is a group under matrix multiplication.
7. Prove that any group with three elements must be isomorphic to $\mathbb{Z}_3$.
8. Let $G =< a >$ be a finite cyclic group of order n. If $m \in \mathbb{Z}$, then prove that $< a^m >=< a^d >$ where $d = gcd(m, n)$.
9. Find the order of $(1,2)(2,3)(3,4)$.
10. Let $\phi : G_1 \to G_2$ be a group homomorphism. Prove that for any integer n and any $a \in G_1$, $\phi(a^n) = \phi(a)^n$ for all $a \in G_1$.

(P.T.O.)

11. Let N be a normal subgroup of G . Prove that the natural projection $\pi : G \to G/N$ defined by $\pi(x) = xN$, for all $x \in G$, is a group homomorphism, and $ker(\pi) = N$.

12. If R is a commutative ring, then show that $a.0 = 0$ for all $a \in R$.

13. Show that $Aut(\mathbb{Z}_n) \cong \mathbb{Z}_n^{\times}$.

14. If $f(x)$ and $g(x)$ are nonzero polynomials in $F[x]$, then prove that their product $f(x)g(x)$ is nonzero and $\deg(f(x)g(x)) = \deg(f(x)) + \deg(g(x))$.

15. For any element $c \in F$, and any positive integer k, show that $x - c \mid x^k - c^k$.

SECTION B: Answer the following questions. Each carries 5 marks (Ceiling 35 Marks)

16. Let n be a positive integer. Then show that congruence class $[a]_n$ has a multiplicative inverse in $\mathbb{Z}_n$ if and only if $(a, n) = 1$.

17. Prove that he groups $\mathbb{R}$ (under addition) and $\mathbb{R}^+$ (under multiplication) are isomorphic.

18. Find HK in $\mathbb{Z}_{16}^{\times}$ if $H =< [3] >$ and $K =< [5] >$.

19. Let G be a group, and let $a \in G$. Then prove that the set $< a >$ is a subgroup of G. Also prove that if K is any subgroup of G such that $a \in K$, then $< a > \subseteq K$.

20. If N is a normal subgroup of G, then prove that the set of left cosets of N forms a group under the coset multiplication given by $aNbN = abN$ for all $a, b \in G$.

21. Write down the formulas for all homomorphisms from $\mathbb{Z}$ onto $\mathbb{Z}_{12}$.

22. Find the subgroup diagram of S_3.

23. Let R be a commutative ring such that $a^2 = a$ for all $a \in R$. Show that $a + a = 0$ for all $a \in R$.

SECTION C: Answer any 2 questions. Each carries 10 marks.

24. State and prove the fundamental theorem on equivalence relations.

25. Show that the smallest order of a nonabelian group is 6.

26. Define the Euler ϕ- function. State and prove a formula for $\phi(n)$.

27. Let G be a group with normal subgroups H, K such that $HK = G$ and $H \cap K = \{e\}$. Show that $H \times K \cong G$.

(2 x 10 = 20 Marks)