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D5BMT1802 (S2) (PAGES 2) Reg. No.............

Name: ....

FIFTH SEMESTER B.Sc. DEGREE EXAMINATION, NOVEMBER 2022

(Supplementary – 2018 Admission)

MATHEMATICS

AMAT5B06T: ABSTRACT ALGEBRA

Time: 3 Hours **Maximum marks: 120**

PART A: Answer *all* the questions. Each carries *one* mark

1. Give an example of a cyclic group having only one generator.
2. Compute the product $(20)(-8)$in $\mathbb{Z}_{26}$.
3. Give an example of group of order 4.
4. Find$\tau\sigma$. If $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 1 & 4 & 5 & 6 & 2 \end{pmatrix}$ and $\tau = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 4 & 1 & 3 & 6 & 5 \end{pmatrix}$.
5. Compute the product of the cycles $(1,4,5)(7,8)(2,5,7)$ that are permutations of $\{1,2,3,4,5,6,7,8\}$
6. What is the order of the cycle $(1,4,5,7)$in S_8.
7. Write all cosets of the subgroup $4\mathbb{Z}$ of $2\mathbb{Z}$.
8. Give an example of homomorphism whose kernal contains only 2 elements.
9. How many group homomorphisms are there of $\mathbb{Z}$onto $\mathbb{Z}$?
10. Write cyclic sub group of $\mathbb{Z}$ generated by 3.
11. Write an integral domain which is not a field
12. Write the field of quotients of $\mathbb{R}$.

(12 x 1= 12 Marks)

PART B: Answer any *ten* questions. Each question carries *4* marks

13. Prove or disprove that $\phi: \langle\mathbb{Q},\cdot\rangle \to \langle\mathbb{Q},\cdot\rangle$ defined by $\phi(x) = x^2$for $x \in \mathbb{Q}$. is an isomorphism
14. Prove or disprove all $n \times n$ diagonal matrices under matrix multiplication is a group.
15. Show that every group G with identity e and such that $x * x = e$ for all $x \in G$ is abelian.
16. Define generater of a group and give an example.
17. Find all orders of subgroups of the group $\mathbb{Z}_6$.
18. Prove or disprove that the function $f_2: \mathbb{R} \to \mathbb{R}$ defined by $f_2(x) = x^2$ is a permutation of $\mathbb{R}$.
19. Find the number of elements in the set $\{\sigma \in S_4 | \sigma(3) = 3\}$
20. Show that any permutation of a finite set of at least two elements is a product of transpositions.
21. Show that if H is a subgroup of index 2 in a finite group G, then every left coset of H is also a right coset of H.
22. Let $\phi: G \to G'$ be a group homomorphism. Show that if $|G|$ is finite, then $|\phi\ [G]|$ is finite and is a divisor of $|G|$
23. Show that an intersection of subrings of a ring R is again a subring of R.
24. Prove or disprove that $n\mathbb{Z}$ has zero divisors if n is not prime.
25. If p is a prime, then show that $\mathbb{Z}_p$, has no divisors of 0.
26. Show that any two fields of quotients of an integral domain D are isomorphic.

(10 x 4 = 40 Marks)

(PTO)

PART C: Answer any *six* questions. Each carries *7* marks

27.(a) If m is a positive integer and n is any integer, then prove that there exist unique integers q and r such that $n = mq + r$ and $0 \leq r < m$.

(b) Find the greatest common divisor of the two integers 32 and 24.

28. (a) Show that if G is a group with binary operation $*$, and if a and b are any elements of G, then the linear equations $a * x = b$ and $y * a = b$ have unique solutions x and y in G.

(b) Let n be a positive integer and let n $\mathbb{Z}$ = {nm|m∈ $\mathbb{Z}$). Show that $(n\mathbb{Z}, +) \simeq (\mathbb{Z}, +)$.

29. (a) Let $2\mathbb{Z} = \{2n|n \in \mathbb{Z}\}$, so that $2\mathbb{Z}$ is the set of all even integers, positive, negative, and zero. Prove or disprove that $\langle \mathbb{Z}, + \rangle$ is isomorphic to $\langle 2\mathbb{Z}, + \rangle$, where + is the usual addition.

(b) Prove or disprove that $\phi: \langle \mathbb{Q}, + \rangle \rightarrow \langle \mathbb{Q}, + \rangle$ defined by $\phi(x) = x/2$for $x \in \mathbb{Q}$. is an isomorphism

30.(a) If R is a ring with additive identity 0. then for any $a, b \in R$ We have

1. $0a = a0 = 0,$
2. $a(-b) = (-a)b = -(ab),$

(b) Prove or disprove every field is also a ring.

31. (a) Show that every field F is an integral domain.

(b) Let R be a ring with unity. If $n \cdot 1 \neq 0$ for all $n \in \mathbb{Z}^+$ then prove that R has characteristic 0. If $n \cdot 1 = 0$ for some $n \in \mathbb{Z}^+$, then also prove that the smallest such integer n is the characteristic of R.

32. Let A be a nonempty set, and let S_A be the collection of all permutations of A. Then prove that S_A is a group under permutation multiplication

33.Let H be a subgroup of a finite group G, Then the order of H is a divisor of the order of G. Prove or disprove the converse

34.Let ϕ be a homomorphism of a group G into a group G'. Show that ϕ preserves the identity element, inverses, and subgroups.

35.(a) Let G be a group and let $a \in G$. Then show that $H = \{a^n | n \in z\}$is a subgroup of G and is the smallest subgroup of G that contains a.

(b) Define generator of a group and give an example.

(6 x7 = 42 Marks)

PART D: Answer any *two* questions. Each question carries *13* marks

36. (a) Show that a subset H of a group G is a subgroup of G if and only if

1. H is closed under the binary operation of G
2. the identity element e of G is in H
3. for all $a \in H$ it is true that $a^{-1} \in H$ also

(b) Show that if H and K are subgroups of an abelian group G, then $\{hk | h \in H \text{ and } k \in K\}$is a subgroup of G.

37. State and prove Cayley's Theorem.

38. (a) Suppose H and K are subgroups of a group G such that $K \leq H \leq G$, and suppose $(H : K)$and$(G : H)$ are both finite. Then prove that $(G : K)$ is finite, and $(G: K) = (G : H)(H : K)$.

(b) Let H be a subgroup of a group G such that $g^{-1}hg \in H$ for all $g \in G$ and all $h \in H$. Show that every left coset gH is the same as the right cosetHg.

(2 x 13 = 26 Marks)