D4BEM2304 (PAGES 3) Reg.No..............................

Name: ....

FOURTH SEMESTER B.Sc. DEGREE EXAMINATION, APRIL 2025

(Regular/Improvement/Supplementary)

ECONOMICS AND MATHEMATICS (DOUBLE MAIN)

GDMT4B04T: ABSTRACT ALGEBRA

Time: 2.5 Hours **Maximum: 80 Marks**

SECTION A: Answer the following questions. Each carries 2 marks. (Ceiling 25 Marks)

1. Find all divisors of zero in $\mathbb{Z}_{12}$.

2. Let A be the set of all integers and let B be the set of all nonzero integers. On the set $S = A \times B$ of ordered pairs, define $(m, n) \sim (p, q)$ if $mq = np$. Show that $\sim$ is an equivalence relation.

3. Find the inverse of $(1, 2, 4, 5, 6)$.

4. Let G be a group. Prove that if $x^2 = e$ for all $x \in G$, then G is abelian.

5. Prove or disprove this statement: $\mathbb{Z}_8^\times$ is cyclic.

6. Give an example of a nonabelian group of order 8.

7. Show that $\mathbb{Z}_5^\times$ is not isomorphic to $\mathbb{Z}_8^\times$ by showing that the first group has an element of order 4 but the second group does not.

8. Let $G =< a >$ be a finite cyclic group of order n. If $m \in \mathbb{Z}$, then prove that $< a^m >=< a^d >$ where $d = gcd(m, n)$

9. Find the subgroup diagram of $\mathbb{Z}_{125}$.

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10. Prove that the set of all even permutations of S_n is a subgroup of S_n.

11. Give a homomorphism from $GL_n(\mathbb{R})$ onto $\mathbb{R}^\times$.

12. Let $\phi : G_1 \to G_2$ be a group homomorphism. Then prove that if $a \in G_1$ and a has order n, then the order of $\phi(a)$ in G_2 is a divisor of n.

13. Let N be a normal subgroup of G, and let $a, b, c, d \in G$. If $aN = cN$ and $bN = dN$, then show that $abN = cdN$.

14. Let R be a commutative ring. Then prove that the set $R^\times$ of units of R an abelian group under the multiplication of R.

15. State second isomorphism theorem

SECTION B: Answer the following questions. Each carries 5 marks. (Ceiling 35 Marks)

16. Let S be a set, and let $\sim$ be an equivalence relation on S. Then prove that each element of S belongs to exactly one of the equivalence classes of S determined by the relation $\sim$.

17. Let G be a group, and let $a \in G$. Then prove that the set $< a >$ is a subgroup of G. Also prove that if K is any subgroup of G such that $a \in K$, then $< a > \subseteq K$.

18. Find HK in $\mathbb{Z}_{16}^\times$ if $H =< [3] >$ and $K =< [5] >$.

19. Prove that the groups $\mathbb{R}$ (under addition) and $\mathbb{R}^+$ (under multiplication) are isomorphic.

20. Find all subgroups of $\mathbb{Z}_{24}$ and draw its subgroup diagram.

21. Show that if G is any group of permutations, then the set of all even permutations in G forms a subgroup of G .

22. Write down the formulas for all homomorphisms from $\mathbb{Z}$ onto $\mathbb{Z}_{12}$.

23. Let $G = S_3$, the group of all permutations on a set with three elements, and let $G = \{e, a, a^2, b, ab, a^2b\}$, where $a^3 = e, b^2 = e$, and $ba = a^2b$. Find all left and right cosets of the subgroup $H = \{e, b\}$.

SECTION C: Answer any two questions. Each carries ten marks.

24. Prove that if a permutation is written as a product of transpositions in two ways, then the number of transpositions is either even in both cases or odd in both cases.

25. Let $S = R - \{-1\}$. Define$*$ on S by $a * b = a + b + ab$. Show that $(S, *)$ is a group

26. Find all subgroups of D_4 and draw its subgroup diagram.

27. Let S be a commutative ring, and let R be a subset of S. Then prove that R is a subring of S if and only if
    (i) R is closed under addition and multiplication;
    (ii) if $a \in R$, then $-a \in R$;
    (iii) R contains the identity of S.

(2 X 10 = 20 Marks)