D3MAH2201 **No.............................**

Name: ....

THIRD SEMESTER UG EXAMINATION, NOVEMBER 2023

GMAH3B10T - REAL ANALYSIS I

Time: 3 hours Maximum Marks: 80

Part A. Answer all the questions. Each question carries 1 mark.

Choose the correct answer.

1. Which one of the following is uncountable?
   A) $\{1,2,3,4\}$ B) $\mathbb{Q}$ C) $\mathbb{R}$ D) $\mathbb{N}$
2. If $a, b \in \mathbb{R}$, then $|a - b| \leq$
   A) $|a| - |b|$ B) $|a| + |b|$ C) $|a + b|$ D) $|ab|$
3. $\lim_{n\to\infty} \sqrt{n+1} - \sqrt{n} =$
   A) ∞ B) 0 C) 1 D) 2
4. The series $\sum_{n=1}^{\infty} \frac{\cos n}{n^2}$ is
   A) Convergent B) Divergent C) Oscillating D) None of these
5. Which of the following is compact
   A) (0, 1) B) $\mathbb{R}$ C) [0, 1] D) [1, ∞)

Fill in the Blanks. Each question carries 1 mark

6. The set A of real numbers x such that $2x + 3 \leq 7$ is -------
7. If $S = \left\{\frac{1}{n};\quad n \in \mathbb{N}\right\}$, then $\inf S =$ ---------
8. The 3-tail of the sequence $X = (2, 4, 6, \dots, 2n, \dots)$ is ---------
9. The series $\sum \frac{1}{n^p}$ is convergent if --------
10. A subset of $\mathbb{R}$ is closed if and only if it contains all of its -----

(10x1=10 Marks)

Part B. Answer any eight questions. Each question carries 2 marks.

11. Prove that $a.a = a$ implies $a = 0$ or $a = 1$.
12. Let $I_n = [0, \frac{1}{n}]$; $n \in \mathbb{N}$. Prove that $\bigcap_{n=1}^{\infty} I_n = \{0\}$.
13. State monotone convergence theorem.
14. If a sequence is convergent, then it is bounded. Is the converse true? Justify your answer.
15. Find $\lim_{n\to\infty} \left(\frac{\sqrt{n}-1}{\sqrt{n}+1}\right)$
16. Prove that the sequence $(1, \frac{1}{2}, 3, \frac{1}{4}, \dots.)$ is divergent.
17. State true or false, 'a Cauchy sequence of real number is unbounded'. Justify with an example.
18. Show that $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n+1}}$ is divergent.
19. Give an example of a set which is neither open nor closed.
20. Exhibit an open cover of the interval $(1, 2]$ that has no finite subcover.

(8x2=16 Marks)

(PTO)

Part C. Answer any six questions. Each question carries 4 marks.

21. Prove that union of countable sets are countable.
22. Determine the set $B = \left\{x \in \mathbb{R};\ \frac{2x+1}{x+2} < 1\right\}$
23. Let (y_n) be a sequence defined by $y_1 = 1,\ y_{n+1} = \frac{1}{4}(2y_n + 3)$. Show that $\lim_{n\to\infty} y_n = \frac{3}{2}$.
24. Prove that a bounded sequence of real numbers has a convergent subsequence.
25. Let $x_n = \begin{cases} \frac{1}{1+n}, & \text{if } n \text{ is odd} \\ \frac{-1}{n}, & \text{if } n \text{ is even} \end{cases}$ be the n^{th} term of the sequence $X = (x_n)$. Find lim sup x_n and lim inf x_n.
26. Discuss the convergence of the series $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$. If convergent, find the limit of the series.
27. Show that
    (a) The union of an arbitrary collection of open subset in $\mathbb{R}$ is open.
    (b) The intersection of an arbitrary collection of closed set in $\mathbb{R}$ is closed.
28. Define compact set. Determine whether the set $(0, 1)$ is compact. Justify ?

(6x4=24 Marks)

Part D. Answer any two questions. Each carries 15 marks.

29. State and prove the denumerability of the following sets,
    (a) The set of rational numbers $\mathbb{Q}$.
    (b) The set $\mathbb{N} \times \mathbb{N}$, where $\mathbb{N}$ is the set of all natural numbers.
30. (a) State and prove Squeeze theorem.
    (b) Using squeeze theorem find the limit of the sequences $\left(\frac{\sin n}{n}\right)$ and $\left(n^{\frac{1}{n^2}}\right)$.
31. (a) State Limit comparison test for the convergence of series and discuss the convergence of $\sum \frac{1}{n^2-n+1}$.
    (b) Prove that a subset $\mathbb{R}$ is closed if and only if it contains all of its cluster points.

(2x15=30 Marks)