D4AMT2303 (2 Pages) Name............................
Reg.No..........................

FOURTH SEMESTER M. Sc. DEGREE EXAMINATION, APRIL 2025
(Regular/Improvement/Supplementary)
MATHEMATICS
FMTH4E10 - DIFFERENTIAL GEOMETRY

Time: 3 Hours **Maximum Weightage: 30**

Part A: Answer *all* questions. Each carries 1 weightage.

1. Sketch the level curves of the function $f(x_1, x_2) = x_1 - x_2$.
2. For what values of c is the level set $f^{-1}(c)$ an n-surface, where f is defined as $f(x_1, x_2, x_3, \dots, x_{n+1}) = x_1^2 + x_2^2 + \cdots + x_{n+1}^2$.
3. Define spherical image of an n-surface.
4. Find the velocity and acceleration of the parametrized curve $\alpha: I \to R^4$ where $I = [0, 2\pi]$ and $\alpha(t) = (\cos t, \sin t, \cos t, \sin t)$.
5. Prove that if $\boldsymbol{X}$ is parallel along a parametrized curve $\alpha: I \to R^{n+1}$, then $\boldsymbol{X}$ has constant length.
6. Compute $\nabla_{\boldsymbol{v}} f$, where $f(x_1 , x_2) = 2x_1^2 + 3x_2^2, \boldsymbol{v} = (1,0,2,1)$.
7. Define circle of curvature of an oriented plane curve.
8. Define normal space to an n $-$surface S in R^{n+k}.

(8 × 1 = 8 weightage)

Part B: Answer any *two* questions *from each unit*. Each carries 2 weightage.

Unit 1

9. Let $\boldsymbol{X}$ be a smooth vector field on an open set $U \subset R^{n+1}$ and let $p \in U$. Prove that there exists an integral curve of $\boldsymbol{X}$ passing through the point p.
10. Show that the maximum and minimum values of the function $g(x_1, x_2) = ax_1^2 + 2bx_1x_2 + cx_2^2$ where $a, b, c \in R$ on the unit circle $x_1^2 + x_2^2 = 1$ are the eigen values of the matrix $\begin{bmatrix} a & b \\ b & c \end{bmatrix}$.
11. Let $S \subset R^{n+1}$ be a connected n-surface in R^{n+1} . Prove that there exist on S exactly two smooth unit normal vector fields $N_1$and N_2 and $N_2(p) = -N_1(p)$ for all $p \varepsilon S$.

Unit 2

12. Prove that for each $a, b, c, d \in R$, the parametrized curve in R^3 defined by $\alpha(t) = (\cos(at + b), \sin(at + b), ct + d)$ is a geodesic on the cylinder $x_1^2 + x_2^2 = 1$.
13. Prove that parallel transport is a vector space isomorphism.
14. Prove that the local parametrizations of plane curves are unique up to reparametrization.

(P.T.O.)

Unit 3

15. Prove that on each compact oriented n-surface in R^{n+1} there exists a point p such the second fundamental form at p is definite.
16. Find the Gaussian curvature of a torus in R^3 .
17. Let $\varphi: U \rightarrow R^{n+1}$ be parametrized n-surface in R^{n+1} and let $p \in U$. Prove that there exists an open set $U_1 \subset U$ about p such that $\varphi(U_1)$ is an n-surface in R^{n+1} .

(6 × 2 = 12 weightage)

Part C: Answer any *two* questions. Each carries 5 weightage.

18. Prove that at each regular point p on a level set $f^{-1}(c)$ of a smooth function, the tangent space equal to $[\nabla f(p)]^{\perp}$.
19. a) Show that the Weingarten map at each point p of an oriented n-surface in R^{n+1} is self-adjoint.
    b) Compute the Weingarten map for the hyperplane $a_1x_1 + a_2x_2 + \cdots + a_{n+1}x_{n+1} = b$ where $(a_1, a_{2,} \ldots a_{n+1}) \neq (0,0, \ldots 0)$.
20. Let C be an oriented plane curve. Prove that there exists a global parametrization of C if and only if C is connected.
21. a) Let S be an n-surface in R^{n+1} and let $p \in S$. Prove that there exists an open set V about p in R^{n+1} and a parametrized n-surface $\varphi: U \rightarrow R^{n+1}$ such that φ is a one to one open map from U onto $V \cap S$.
    b) State and prove the inverse function theorem for n-surfaces.

(2 × 5 = 10 weightage)