D4AMT2301 (2 Pages) Name............................
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FOURTH SEMESTER M. Sc. DEGREE EXAMINATION, APRIL 2025
(Regular/Improvement/Supplementary)
MATHEMATICS
FMTH4C15 - ADVANCED FUNCTIONAL ANALYSIS

Time: 3 Hours **Maximum Weightage: 30**

Part A: Answer *all* questions. Each carries 1 weightage.

1. Show that the spectrum of a bounded operator on a Banach space X is a closed set.
2. Show that the identity operator on an infinite dimensional Banach space X is not compact.
3. Show that if L is invariant with respect to a symmetric operator A, then so is $L^{\perp}$.
4. If $A \geq 0$, $B \geq 0$ and $AB = BA$, then prove that $AB \geq 0$.
5. Prove that every ortho projection P satisfies $0 \leq P \leq I$.
6. State Hilbert Theorem on the spectral decomposition of self adjoint bounded operators.
7. State and prove bounded inverse theorem.
8. Prove that an element 'x' is invertible if and only if x does not belong to any maximal ideal.

(8 × 1 = 8 weightage)

Part B: Answer any *two* questions from each unit. Each carries 2 weightage.

Unit 1

9. Let X be a Banach space and T be a compact operator on X. If the operator $T - \lambda I$ is onto, where $\lambda \neq 0$, prove that $\ker T_{\lambda} \neq \{0\}$.
10. Find the spectrum of the right shift operator on l_2.
11. Let A be an operator on a Hilbert space H. Prove that $< Ax, x > \in R$ for any $x \in H$ if and only if A is symmetric.

Unit 2

12. State and prove Generalized Cauchy Schwartz Inequality.
13. If P_n be a sequence of ortho projections converging strongly to an operator P. Then prove that P is an ortho projection.

(P.T.O.)

14. Explain the construction of the spectral integral of a bounded self adjoint operator in a Hilbert space H.

Unit 3

15. Let X and Y be Banach spaces. Let $A : X \to Y$ be a closed graph operator and Dom A = X. Show that A is continous.

16. State and prove Hahn Banach theorem in complex case.

17. If μ is perfectly convex function on a Banach space X such that $\mu(\lambda x) = \lambda\mu(x)$ for any $\lambda \geq 0$, show that μ is a continous function.

(6 × 2 = 12 weightage)

Part C: Answer any *two* questions. Each carries 5 weightage.

18. State and prove the first Hilbert Schmidt theorem.

19. (a). If $A_0 \leq A_1 \leq A_2 \ldots\ldots\ldots \leq A_n \leq A$, show that there exists a bounded operator B such that B and $A_n x \to Bx$ for all $x \in H$.

    (b) If P is a projection and $\operatorname{Im} P \perp \operatorname{Ker} P$ then prove that $P = P^*$.

20. (a). If X and Y are Banach spaces and if $A : X \to Y$ is a bounded linear operator on to Y, then show that A is an open map.

    (b) Show that every basis of a Banach space is a Schauder basis.

21. (a). Prove that the set of all invertible elements of a Banach Algebra is an open subset of A.

    (b) State and prove Gelfand theorem on maximal ideal.

(2 × 5 = 10 weightage)