D4AMT2203 (3 Pages) Name............................

Reg.No..........................

FOURTH SEMESTER M. Sc. DEGREE EXAMINATION, APRIL 2024

(Regular/Improvement/Supplementary)

MATHEMATICS

FMTH4E10 - DIFFERENTIAL GEOMETRY

Time: 3 Hours **Maximum Weightage: 30**

Part A: Answer *all* questions. Each carries 1 weightage.

1. Find the integral curve through the point $p = (1,1)$ of the vector field $\mathbb{X}(p) = (p, X(p))$, where $X(p) = (0,1)$ on $\mathbb{R}^2$.
2. Let $f: U \to \mathbb{R}$ be a smooth function and let $\alpha: I \to U$ be an integral curve of ∇f. Show that $\left(\frac{d}{dt}\right)(f \circ \alpha)(t) = \|\nabla f(\alpha(t))\|^2$ for all $t \in I$.
3. For what values of c is the level set $f^{-1}(c)$ an n-surface, where $f(x_1, \dots, x_{n+1}) = {x_1}^2 + \cdots + {x_{n+1}}^2$?
4. Prove that geodesics have constant speed.
5. Prove that the velocity vector field along a parameterized curve α in S is parallel if and only if α is geodesic.
6. Prove that the covariant derivative $\mathbb{X}'(t)$ of a smooth vector field $\mathbb{X}$ is independent of the orientation.
7. Define differential of φ and find the formula for evaluating it.
8. Define chart and give an example.

(8 × 1 = 8 weightage)

Part B: Answer any *two* questions from each unit. Each carries 2 weightage.

Unit 1

9. Define level set and graph of a function $f: U \to \mathbb{R}$, $U \subset \mathbb{R}^{n+1}$. Show that the graph of any function $f: \mathbb{R}^n \to \mathbb{R}$, is a level set for some function $F: \mathbb{R}^{n+1} \to \mathbb{R}$.

(P.T.O.)

10. Define a vector field on $\mathbb{R}^{n+1}$. Sketch the vector field $\mathbb{X}(p) = (p, X(p))$ on $\mathbb{R}^2$ where $X(x_1,\ x_2) = (-x_1, -x_2)$.

11. What do you mean by a vector at a point p tangent to a level set? Show that the gradient of f at $p \in f^{-1}(c)$ is orthogonal to all vectors tangent to $f^{-1}(c)$.

Unit 2

12. Show that the Weingarten map of an n-surface S at a point p in S is self-adjoint.

13. Let C be a plane curve oriented by the unit normal vector field $\mathbb{N}$. Let $\alpha: I \to C$ be a unit speed local parameterization of C. For $t \in I$, $\mathbb{T}(t) = \dot{\alpha}(t)$, show that $\dot{\mathbb{T}} = (\kappa \circ \alpha)(\mathbb{N} \circ \alpha)$ and $(\mathbb{N} \dot{\circ} \alpha) = -(\kappa \circ \alpha)\mathbb{T}$.

14. Let C be a connected oriented plane curve and let $\beta: I \to C$ be a unit speed global parameterization of C. Then show that β is either one to one or periodic. Show further that β is periodic if and only if C is compact.

Unit 3

15. Let V be a finite dimensional vector space with dot product and let $L: V \to V$ be a self-disjoint linear transformation on V. Then prove that there exists an orthonormal basis for V consisting of eigenvectors of L.

16. Find the Gaussian curvature $K: S \to \mathbb{R}$, where S is the ellipsoid $\frac{x_1^2}{a^2} + \frac{x_2^2}{b^2} + \frac{x_3^2}{c^2} = 1$, ($a, b$ and c all $\neq 0$).

17. Show for a parameterized n-surface $\varphi: U \to \mathbb{R}^{n+1}$ in $\mathbb{R}^{n+1}$ and for $p \in U$, there exists an open set $U_1 \subset U$ about p such that $\varphi(U_1)$ is an n-surface in $\mathbb{R}^{n+1}$.

(6 × 2 = 12 weightage)

Part C: Answer any *two* questions. Each carries 5 weightage.

18. (a) Let S be an n-surface in $\mathbb{R}^{n+1}$, $S = f^{-1}(c)$, where $f: U \to R$ is such that $\nabla f(q) \neq 0$ for all $q \in S$. Suppose $g: U \to R$ is a smooth function and $p \in S$ is an extreme point of g on S. Prove that there exists a real number λ such that $\nabla g(p) = \lambda \nabla f(p)$.

(b) What do you mean by an oriented n-surface S in $\mathbb{R}^{n+1}$? Show that on a connected n-surface S in $\mathbb{R}^{n+1}$ there exists always exactly two orientations.

19. Let S be a compact connected oriented n-surface in $\mathbb{R}^{n+1}$ exhibited as a level set $f^{-1}(c)$ of a smooth function $f: \mathbb{R}^{n+1} \to \mathbb{R}$ with $\nabla f(p) \neq 0$ for all $p \in S$. Then show that the Gauss map maps S onto the unit sphere S^n.

20. Let S be an n-surface in $\mathbb{R}^{n+1}$, let $p \in S$ and let $\mathbb{v} \in S_p$. Then prove there exists an open interval I containing 0 and a geodesic $\alpha: I \to S$ such that:

    (i) $\alpha(0) = p$ and $\dot{\alpha}(0) = \mathbb{v}$.

    (ii) If $\beta: \hat{I} \to S$ is any other geodesic in S with $\beta(0) = p$ and $\dot{\beta}(0) = \mathbb{v}$ then $\hat{I} \subset I$ and $\beta(t) = \alpha(t)$ for all $t \in \hat{I}$.

21. (a) Let S be a compact oriented n-surface in $\mathbb{R}^{n+1}$. Prove that there exists a point p on S such that the second fundamental form of S at p is definite.

    (b) State and prove the Inverse function theorem for n-surfaces.

(2 × 5 = 10 weightage)