D4AMT2202

Name ....

Reg. No. ....

FOURTH SEMESTER M. Sc. DEGREE EXAMINATION, APRIL 2024

(Regular/Improvement/Supplementary)

MATHEMATICS

FMTH4E08 - ALGEBRAIC TOPOLOGY

Time: 3 Hours

Maximum Weightage: 30

Part A: Answer *all* questions. Each carries 1 weightage.

1. Define a geometric complex with an example.
2. Let $S = \{(1, 3, 7), (2, 7, 6.5), (0, 4, 7)\}$. Determine whether S is geometrically independent.
3. Define a k-simplex. What are the barycentric coordinates of the point $(5, 3)$ determined by geometrically the independent set $\{(1, 7), (3, 0), (9, 3)\}$?
4. Let K be an oriented simplicial complex K. Define the chain group $C_p(K)$.
5. Let K be the closure of a 2-simplex $\langle a_0a_1a_2\rangle$ with orientation induced by the ordering $a_0 < a_1 < a_2$. Prove that $Z_0(K)$ is isomorphic to the group $\mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$.
6. State the Euler-Poincaré Theorem.
7. Prove that any triangulation of S^2 must have at least four vertices, at least six 1-simplexes, and at least four 2-simplexes.
8. Let X be a topological space and let $x_0 \in X$. Define the fundamental group $\pi_1(X, x_0)$.

(8 x 1= 8 weightage)

Part B: Answer *any two* questions from *each unit.* Each carries 2 weightage.

Unit 1

9. Prove that a set of $k + 1$ points in $\mathbb{R}^n$ is geometrically independent if and only if no $p + 1$ of the points lie in a hyperplane of dimension less than or equal to $p - 1$.
10. Prove that a set $A = \{a_0, a_1, \ldots, a_k\}$ of points in $\mathbb{R}^n$ is geometrically independent if and only if the set of vectors $\{a_1 - a_0, \ldots, a_k - a_0\}$ is linearly independent.
11. Let X be a topological space. Define a triangulation of X. Prove that every simplex is a convex set.

Unit 2

12. If K is an oriented complex, then prove that $B_p(K) \subset Z_p(K)$ for each integer p such that $0 \leq p \leq n$, where n is the dimension of K.
13. If S is a simple polyhedron with V vertices, E edges, and F faces, then prove that $V - E + F = 2$.
14. Prove that the n-sphere S^n is not contractible for any $n \geq 0$.

Unit 3

15. Let X be a topological space. Prove that the equivalence of paths is an equivalence relation on the set of paths in a space X.
16. Prove that the fundamental group $\pi_1\left(S^1\right)$ is isomorphic to the group $\mathbb{Z}$ of integers under addition.
17. Give an example of a simply connected space that is not contractible.

(6 x 2= 12 weightage)

(P.T.O.)

Part C: Answer any *two* questions. Each carries 5 weightage.

18. (a) If $\sigma : I \rightarrow S^1$ is a path in S^1 with initial point 1, then prove that there is a unique covering path $\tilde{\sigma} : I \rightarrow \mathbb{R}$ with initial point 0.

    (b) If K is an oriented complex and $p \geq 2$, then prove that the composition $\partial\partial : C_p(K) \rightarrow C_{p-2}(K)$ in the diagram

$$C_p(K) \overset{\partial}{\rightarrow} C_{p-1}(K) \overset{\partial}{\rightarrow} C_{p-2}(K)$$

    is the trivial homomorphism.

19. (a) Let K be a geometric complex with two orientations, and let K_1, K_2 denote the resulting oriented geometric complexes. Then prove that the homology groups $H_p(K_1)$ and $H_p(K_2)$ are isomorphic for each dimension p.

    (b) Let $|K|$ and $|L|$ be polyhedra with triangulations K and L respectively and $f : |K| \rightarrow |L|$ a continuous function. Prove that there is a barycentric subdivision $K^{(k)}$ of K and a continuous function $g : |K| \rightarrow |L|$ such that (a) g is a simplicial map from $K^{(k)}$ into L, and (b) g is homotopic to f.

20. (a) Prove that there is a vector field on $S^n, n \geq 1$, if and only if n is odd.

    (b) If two continuous maps $f, g : S^n \rightarrow S^n$ are homotopic, then prove that they have the same degree.

21. (a) Let K be an oriented complex, σ^p an oriented p-simplex of K and $\sigma^{p-2}a(p-2)$-face of σ^p. Then prove that

$$\sum \left[\sigma^p, \sigma^{p-1}\right]\left[\sigma^{p-1}, \sigma^{p-2}\right] = 0, \quad \sigma^{p-1} \in K$$

    (b) For any complex K prove that $\lim_{s\rightarrow\infty}$ mesh $K^{(s)} = 0$

(2 x 5= 10 weightage)