D4AMT2201 (2 Pages) Name:.........................
Reg. No:.....................

FOURTH SEMESTER M.Sc. DEGREE EXAMINATION, APRIL 2024

(Regular/Improvement/Supplementary)

MATHEMATICS

FMTH4C15 - ADVANCED FUNCTIONAL ANALYSIS

Time : 3 Hours Maximum Weightage : 30

Part A: *Short answer questions. Answer* **all** *questions. Each carries* ***1*** *weightage.*

1. Define point spectrum,continuous spectrum and residual spectrum of a bounded operator A.
2. Let H be a Hilbert space and A be a bounded operator on H such that $\langle Ax, x\rangle \in \mathbb{R}$ for every $x \in H$. Prove that A is a symmetric operator.
3. State Minimax Principle.
4. If C is symmetric operator and $A \leq B$, then prove that $A + C \leq B + C$
5. Prove that every orthoprojection P satisfies $0 \leq P \leq I$
6. Define perfectly convex set. Prove that $K = [0, 1]$ is perfectly convex.
7. Prove that the linear operator A admits a closure if and only if $Ax_n \to y$ and $x_n \to 0$ imply $y = 0$.
8. Define Banach Algebra with an example

(8 x 1= 8 weightage)

Part B

Answer any **two** *questions from* **each unit.** *Each carries* ***2*** *weightage.*

UNIT I

9. Prove that for every $\varepsilon > 0$, there is only a finite number of linearly independent eigen vectors corresponding to eigen values λ_i with $|\lambda_i| \geq \varepsilon$.

10. State and Prove Fredholm's second theorem.

11. Define $C = \sup_{x \neq 0} |\langle Ax, x\rangle|/||x||^2$. If A is a symmetric operator then prove that $C = ||A||$.

UNIT II

12. State and prove Generalized Cauchy- Schwartz inequality.

13. Let $P_1P_2 = P_2P_1 = P$. Then P is an orthoprojection and $E = ImP = E_1 \cap E_2 \quad (E_1 = P_iH)$

14. Let $Q_n(t)$ and $P_n(t)$ be sequences of polynomials. Assume that for all $t \in [m, M]$, $Q_n(t) \to \psi(t)$ in K and $P_n(t) \to \phi(t)$ in K. Let $\psi(t) \leq \phi(t)$ for all $t \in [m, M]$. Then prove that $\lim\limits_{n\to\infty} Q_n(A) =: B_1 \leq B_2 := \lim\limits_{n\to\infty} P_n(A)$.

UNIT III

15. State and prove Baire-Category Theorem

16. State and prove closed graph theorem

17. Prove that if $||x|| < 1$, then $e - x$ is invertible and $(e - x)^{-1} = \sum_{k=0}^{\infty} x^k$

(6 x 2= 12 weightage)

Part C: *Answer any* **two** *questions. Each carries* **5** *weightage.*

18. (a) Consider the shift operator $Ae_i = e_{i+1}$ on l_2. Find $\sigma(A), \sigma_c(A), \sigma_r(A)$.

    (b) Let T be a compact operator prove that $\Delta_\lambda = X$ implies $\ker T_\lambda = 0$

19. State and Prove the first Hilbert Schmidt theorem.

20. Let A be such that $mI \leq A \leq MI$ for some $m, M \in \mathbb{R}$ and let P be a polynomial satisfying $P(z) \geq 0$ for all $z \in [m, M]$. Then Prove that $P(A) \geq 0$.

21. Prove that every basis of a Banach space is a Schauder basis.

(2 x 5= 10 weightage)