D4AMT2003 **(2 Pages)** **Name............................**
Reg.No.........................

FOURTH SEMESTER M. Sc. DEGREE EXAMINATION, APRIL 2022
(Regular/Improvement/Supplementary)

MATHEMATICS
FMTH4E10 - DIFFERENTIAL GEOMETRY

Time: 3 Hours **Maximum Weightage: 30**

Part A: Answer *all* questions. Each carries *one* weightage.

1. Express the graph of the function $f: R^2 \to R$ defined by $f(x_1, x_2) = x_1 - x_2$ as the level set of some function $F: R^3 \to R$.
2. Find and sketch the gradient field of the function $f(x_1, x_2) = {x_1}^2 + {x_2}^2$.
3. Show that the set S of all unit vectors at all points of R^2 forms a 3-surface in R^4.
4. Find the speed of the parametrizedcurve $\alpha(t) = (t, t^2)$.
5. Compute $\nabla_v f$ where $f(x_1, x_2) = 2{x_1}^2 + 3{x_2}^2$ and $\boldsymbol{v} = (1,0,2,1)$.
6. Find the arclength of the parametrized curve $\alpha(t) = (\cos 3t, \sin 3t,\ 4t), I = [-1,1]$.
7. Define the second fundamental form of an oriented n-surface.
8. Let $\phi: U_1 \to U_2$ and $\varphi: U_2 \to R^k$ be smooth, where $U_1 \subset R^n$ and $U_2 \subset R^m$. Verify the Chain rule $d(\varphi o \phi) = d\varphi \ o \ d\phi)$.

(8 × 1 = 8 weightage)

Part B: Answer any *two* questions from each unit. Each carries *two* weightage

Unit 1

9. Find the integral curve through $p = (a, b)$ of the vector field $\boldsymbol{X}(x_{1,} x_2) = (x_{1,}\ x_{2,}\ {}_{-}x_{2,} x_1)$.
10. Let S be a n-surface in R^{n+1}, $S = f^{-1}(c)$ where $f: U \to R$ is such that $\nabla f(q) \neq 0$ for all $q \in S$. Suppose $g: U \to R$ is a smooth function and $p \in S$ is an extreme point of g on S, then prove that there exists a real number λ such that $\nabla g(p) = \lambda \nabla f(p)$.
11. Let $S \subset R^{n+1}$ be a connected n-surface in R^{n+1}. Prove that there exist on S exactly two smooth unit normal vector fields N_1 and N_2, and that $N_2(p) = -N_2(p)$ for all p $\in$ S.

Unit 2

12. Let **X** and **Y** be smooth vector fields tangent to the n-surface S along the parametrized curve $\alpha: I \to R^{n+1}$. Verify that $(\boldsymbol{X}.\boldsymbol{Y})' = \boldsymbol{X}'.\boldsymbol{Y} + \boldsymbol{X}.\boldsymbol{Y}'$.

(P.T.O.)

13. Show that the Weingarten map at each point p of an oriented n-surface in R^{n+1} is self adjoint.

14. Prove that the curvature of the circle $f^{-1}(r^2)$, where $f(x_1, x_2) = (x_1 - a)^2 + (x_2 - b)^2$ oriented by the outward normal $\frac{\nabla f}{||\nabla f||}$ is $\frac{-1}{r}$.

Unit 3

15. Prove that on each compact oriented n-surface S in R^{n+1}, there exists a point p such that the second fundamental form of at p is definite.

16. Show that the normal curvature of the sphere $x_1^2 + x_2^2 + \cdots + x_{n+1}^2 = r^2 (\boldsymbol{N} = \frac{-\nabla f}{||\nabla f||})$ at any point p in any direction is the constant $\frac{1}{r}$.

17. Let S be a compact, connected oriented n-surface in R^{n+1} whose Gauss Kronecker curvature is nowhere zero. Prove that the Gauss map $N: S \to S^n$ is a diffeomorphism.

(6 × 2 = 12 weightage)

Part C: Answer any *two* questions. Each carries 5 weightage.

18. a) Let S be a compact connected oriented n-surface in R^{n+1} exhibited as a level set $f^{-1}(c)$ of a smooth function $f: R^{n+1} \to R$ with $\nabla f(p) \neq 0$ for all $p \in S$. Prove that the Gauss map maps S onto the unit sphere S^n.

b) Find the spherical image of the hyperboloid ${x_1}^2 - {x_2}^2 - {x_3}^2 = 4, x_1 > 0$ oriented by $N = \frac{\nabla f}{||\nabla f||}$ where $f(x_1, x_2, x_3) = {x_1}^2 - {x_2}^2 - {x_3}^2$.

19. a) Let S be an n-surface in R^{n+1}, let $p \in S$ and $\boldsymbol{v} \in S_P$. Then prove that there exists a maximal geodesic in S passing through p with initial velocity $\boldsymbol{v}$.

b) Prove that for each $a, b, c, d \in R$, the parametrised curve in R^3 defined by

$\alpha(t) = (\sin at + b, ct + d, ct + d)$ is a geodesic in the cylinder ${x_1}^2 + {x_2}^2 = 1$ in R^3.

20. a) Let C be a connected oriented plane curve and let $\beta: I \to C$ be a unit speedparametrization of C. Prove that β is either one one or periodic.

b) Let C be the circle $f^{-1}(r^2)$ where $f(x_1, x_2) = (x_1 - a)^2 + (x_2 - b)^2$ oriented by outward normal. Find a global parametrization of C.

21 a) Let S be an n-surface in R^{n+1} and let $f: S \to R^k$. Then prove that f is smooth if and only $fo\varphi: U \to R^k$ is smooth for each local parametrization $\varphi: U \to S$.

b) State and prove the inverse function theorem for n-surfaces.

(2 × 5 = 10 weightage)