D4AMT2002 (2 Pages) Name............................
Reg.No..........................

FOURTH SEMESTER M. Sc. DEGREE EXAMINATION, APRIL 2022
(Regular/Improvement/Supplementary)

MATHEMATICS
FMTH4E08 - ALGEBRAIC TOPOLOGY

Time: 3 Hours **Maximum Weightage: 30**

Part A: Answer *all* questions. Each carries *one* weightage.

1. Show that the barycentric coordinates of each point in a simplex are unique.
2. Define a *k*-simplex. Give an example of a 2-simplex.
3. Prove that the diameter of a simplex of positive dimension is the length of its longest 1-face.
4. Define the fundamental group of a topological space.
5. How many faces does an n-simplex have?
6. Define a chain mapping.
7. Give two examples of spaces with trivial fundamental group.
8. Define a contractible space. Give an example of a contractible space.

(8 × 1 = 8 weightage)

Part B: Answer any *two* questions from *each unit*. Each carries *two* weightage

Unit 1

9. Let K be an oriented complex, σ^p an oriented p-simplex of K and σ^{p-2} a $(p-2)$ face of σ^p. Then prove that
$\sum[\sigma^p, \sigma^{p-1}][\sigma^{p-1}, \sigma^{p-2}] = 0$ for all σ^{p-1}.
10. Let K be a geometric complex with two orientations, and let K_1, K_2 denote the resulting oriented geometric complexes. Then prove that the homology groups $H_p(K_l)$ and $H_p(K_2)$ are isomorphic for each dimension p.
11. Suppose that K_1 and K_2 are two triangulations of the same polyhedron. Are the chain groups $C_p(K_1)$ and $C_p(K_2)$ isomorphic? Explain.

Unit 2

12. State and prove the Brouwer Fixed Point Theorem.
13. Prove that there is a vector field on S^n, $n \geq 1$, if and only if n is odd.
14. Define mesh of a complex. For any complex K, prove that
$\lim_{s\to\infty} \text{mesh}(K^s) = 0$.

(P.T.O.)

Unit 3

15. Prove that the fundamental group of S^1 is isomorphic to the additive group of integers.
16. If A is a deformation retract of a space X and x_0 is a point of A, then prove that $\pi_1(X, x_0)$ is isomorphic to $\pi_1(A, x_o)$.
17. Let X and Y be spaces with points x_o in X and y_o in Y. Then prove that $\pi_1(X \times Y, (x_0, y_0)) \cong \pi_1(X, x_o) \oplus \pi_1(Y, y_0)$.

(6 × 2 = 12 weightage)

Part C: Answer any *two* questions. Each carries *five* weightage.

18. (a) Let K be a complex with r combinatorial components. Then prove that $H_0(K)$ is isomorphic to the direct sum of r copies of the group $\mathbb{Z}$ of integers.
    (b) If S is a simple polyhedron with V vertices, E edges, and F faces, then prove that $V - E + F = 2$.
    (c) State and prove the Continuity Lemma.
19. (a) Prove that a set of $k + 1$ points in $\mathbb{R}^n$ is geometrically independent if and only if no $p + 1$ of the points lie in a hyperplane of dimension less than or equal to $p - 1$.
    (b) Let K be a 2-pseudomanifold with α_0 vertices, α_1 1-simplexes, and α_2 2-simplexes. Then prove that
    (i) $3\alpha_2 = 2\alpha_1$
    (ii) $\alpha_1 = 3(\alpha_0 - \chi(K))$
    (iii) $\alpha_0 \geq ½ [49 - 24\chi(K)]$
    (c) Using (b) prove that any triangulation of the projective plane P must have at least six vertices, fifteen 1-simplexes, and ten 2-simplexes.
    (d) Prove that the fundamental group $\pi_1(S^1)$ is isomorphic to the group $\mathbb{Z}$ of integers under addition.
20. (a) If K is an oriented complex, then prove that $B_p(K) \subset Zp(K)$ for each integer p such that $0 \leq p \leq n;$ n, where n is the dimension of K.
    (b) Let K and L be simplicial complexes. Prove that every simplicial mapping $p : |K| \to |L|$ is continuous.
    (c) Consider the family $\pi_1(X, x_0)$ of loops in X with base point x_0. Prove that $\pi_1(X, x_0)$ is a group under the following operation:
    $[\alpha] \circ [\beta] := [\alpha * \beta]$ for all $[\alpha], [\beta] \in \pi_1(X, x_0)$.
21. (a) Prove that two loops α and β in S^1 with base point 1 are equivalent if and only if they have the same degree.
    (b) State and prove Euler Poincare Theorem.
    (c) If $\sigma : I \to S^1$ is a path in S^1 with initial point 1, then there is a unique covering path $\sigma^{\sim} : I \to \mathbb{R}$ with initial point 0.

(2 × 5 = 10 weightage)