Name............................

Reg.No..........................

**FOURTH SEMESTER M. Sc. DEGREE EXAMINATION, APRIL 2022**
**(Regular/Improvement/Supplementary)**

**MATHEMATICS**
**FMTH4C15- ADVANCED FUNCTIONAL ANALYSIS**

**Time: 3 Hours** **Maximum Weightage: 30**

**Part A: Answer *all* questions. Each carries *one* weightage.**

1. Define the point spectrum, continuous spectrum and residual spectrum of an operator on a Banach space.
2. Prove that, if $L$ is invariant with respect to a symmetric operator $A$, then so is $L^{\perp}$.
3. If $A$ is a compact self-adjoint operator, then show that $A$ has an eigen value $\lambda$ such that $|\lambda| = \|A\|$.
4. If $A$ is a non negative operator and if $\langle Ax, x\rangle = 0$, then prove that $Ax = 0$.
5. If $E$ is a linear space, $P: E \mapsto E$ is a projection, and if $E_1 = \operatorname{Im} P$ and $E_2 = \ker P$, then show that $E_1 + E_2 = E$ and $E_1 \cap E_2 = 0$.
6. If $P$ is a projection and $\operatorname{Im} P \perp \ker P$, then show that $P = P^*$.
7. For every $x_1 \neq x_2$ in a normed space $X$, prove that there exists $f \in X^*$ such that $f(x_1) \neq f(x_2)$.
8. Define Banach algebras. Give one example.

**(8 × 1 = 8 weightage)**

**Part B: Answer any *two* questions from each unit. Each carries *two* weightage.**

**Unit 1**

9. If $T$ is a compact operator on an infinite dimensional Banach space $X$, then for every $\varepsilon > 0$, prove that there is only a finite number of linearly independent eigenvectors corresponding to eigenvalues $\lambda_i$ with $|\lambda_i| \geq \varepsilon$.
10. If $T$ is a compact operator and $\lambda$ is an eigenvalue of $T$, prove that $\ker T_{\bar{\lambda}}^* \neq 0$ if and only if $\ker T_\lambda \neq 0$.
11. If $T$ is a compact self-adjoint operator on an infinite dimensional Hilbert space $H$, then prove that $\langle Tx, x\rangle \geq 0$ for every $x \in H$ if and only if there are no negative eigen values.

## Unit 2

12. If $A_0 \le A_1 \le \cdots \le A_n \le \cdots \le A,$ show that there exists a bounded operator $B$ and $A_n x \to Bx$ for all $x \in H.$

13. Let $T : E \mapsto E$ be any linear operator, $E_1 + E_2 = E$ and let $P$ be the projection onto $E_1$ parallel to $E_2$. Then prove that $PT = TP$ if and only if $E_1$ and $E_2$ are invariant subspaces of $T.$

14. Let $Q_n(t)$ and $P_n(t)$ be sequences of polynomials such that for all $t \in [m, M],$ $Q_n(t) \downarrow \psi(t) \in K$ and $P_n(t) \downarrow \varphi(t) \in K.$ Let $\psi(t) \le \varphi(t)$ for all $t \in [m, M].$ Then show that $\lim_{n\to\infty} Q_n(A) \le \lim_{n\to\infty} P_n(A).$

## Unit 3

15. Show that every complete metric space is a set of second category.
16. State and prove the Banach open mapping theorem.
17. For a real Banach space $X,$ show that the unit ball $\{f \in X^* : \| f \| \le 1\}$ is a compact set in the $\omega^*-$topology.

**(6 × 2 = 12 weightage)**

**Part C: Answer any *two* questions. Each carries 5 weightage.**

18. (a) If $T$ is a compact operator on a Banach space, prove that $\sigma_p(T) \setminus \{0\} = \overline{\sigma_p(T^*)} \setminus \{0\}.$

    (b) Prove that $\langle Ax, x \rangle \in R$ for any $x \in H$ if and only if $A$ is symmetric.

19. (a) Let $A$ be such that $m \cdot I \le A \le M \cdot I$ for some $m, M \in R$ and let $P$ be a polynomial satisfying $P(z) \ge 0$ for all $z \in [m, M].$ Then show that $P(A) \ge 0.$

    (b) Prove that every orthoprojection $P$ in a Hilbert space satisfies $0 \le P \le I.$

20. (a) State and prove the closed graph theorem.

    (b) State and prove Banach-Steinhaus theorem.

21. (a) Let $p(x)$ be a convex function and $p(x) < \infty$ for all $x \in L.$ Let $f_0$ be a linear functional defined on a subspace $L_0$ of $L$ such that $|f_0(x)| \le p(x)$ for all $x \in L_0.$ Show that there exists a linear functional $f$ on $L$ such that $f|_{L_0} = f_0$ and $|f(x)| \le p(x)$ for every $x \in L.$

    (b) If $X$ is a reflexive space, then show that every closed subspace $E$ of $X$ is also reflexive.

**(2 × 5 = 10 weightage)**