D3AMT2403 **(2 Pages)** **Name............................**
Reg.No..........................

THIRD SEMESTER M. Sc. DEGREE EXAMINATION, NOVEMBER 2025
(Regular/Improvement/Supplementary)

MATHEMATICS
FMTH3C13- FUNCTIONAL ANALYSIS

Time: 3 Hours **Maximum Weightage: 30**

Part A: Answer *all* questions. Each carries 1 weightage.

1. State and prove Minkowski's inequality for integrable functions.
2. Prove that $l_q \subset l_p$, where $1 \leq q \leq p$.
3. Prove that two cosets of a quotient space are either coincide or\ disjoint.
4. Prove that the inner product is a continuous function with respect to both variables.
5. If L is a closed subspace of a Hilbert space, then prove that $(L^{\perp})^{\perp} = L$.
6. Prove that $\operatorname{codim} \ker f = 1$, where $f \in E^{\#}\backslash\{0\}$.
7. Prove or disprove: Strong convergence is weaker than norm convergence.
8. Let $T: L_p(-\infty \to \infty) \mapsto L_p(-\infty \to \infty), 1 \leq p \leq \infty$ with $(Tf)(t) = f(t+1)$. Find T^*.

(8 × 1 = 8 weightage)

Part B: Answer *any two* questions *from each unit*. Each carries 2 weightage.

Unit 1

9. Prove that the complement of an open set is closed and vice versa.
10. Show that every normed space has a completion.
11. Prove that the dimension of $E\backslash E_1$ is n if and only if there exists $x_1, x_2, \cdots x_n$ linearly independent vectors relative to E_1 such that for every $x \in E$, there exists a unique set of numbers $a_1, a_2, \cdots a_n$ and a unique vector $y \in E_1$ such that $x = \sum_{i=1}^{n} a_i x_i + y$. Also prove that $\{[x_1], [x_2], \cdots [x_n]\}$ is a basis of E/E_1.

Unit 2

12. Explain the concept of complete orthonormal system with an example. Prove or disprove that every complete orthonormal system of a Hilbert space is a basis.
13. Prove that $\{f_i\}$ is a complete system if and only if $x \perp f_i, \forall i$ implies $x = 0$, where $\{f_i\}$ is a system of vectors in a Hilbert space H.

(P.T.O.)

14. Let $\{e_i\}$ be an orthonormal system in a Hilbert space H. Prove that $\forall x \in$ H $\sum_{i\geq 1}|\langle x, e_i\rangle|^2 \leq \|x\|^2$. Also prove that there exists a $y \in$ H, $y = \sum_{i=1}^{\infty}\langle x, e_i\rangle e_i$.

Unit 3

15. Define dual operator. Prove that dual of a compact operator is compact.

16. True or False: "For normed spaces X, Y, $L(X \mapsto Y)$ is always a Banach space." Justify your answer.

17. Prove that c_0 is not a reflexive space.

(6 × 2 = 12 weightage)

Part C: Answer any *two* questions. Each carries 5 weightage.

18. Prove that $l_p, 1 \leq p \leq \infty$ is Banach.

19. State and prove the theorem on the space of all bounded linear functionals on a Hilbert space.

20. Prove that M is relatively compact if and only if for every $\epsilon > 0$, there exists a finite ϵ −net in M.

21. State Hahn-Banach theorem. Prove that dual of a normed space is a total set.

(2 × 5 = 10 weightage)