D3AMT2401 **(2 Pages)** **Name.....**
Reg.No.....

THIRD SEMESTER M. Sc. DEGREE EXAMINATION, NOVEMBER 2025
(Regular/Improvement/Supplementary)

MATHEMATICS
FMTH3C11- MULTIVARIABLE CALCULUS AND GEOMETRY

Time: 3 Hours **Maximum Weightage: 30**

Part A: Answer *all* questions. Each carries 1 weightage.

1. Prove that a linear operator A on a finite dimensional vector space X is one-to- one if and only if the range of A is all of X.
2. Prove that to every $A \in L(\mathbb{R}^n, \mathbb{R}^1)$ corresponds a unique $y \in \mathbb{R}^n$ such that $Ax = x \cdot y$. Also prove that $\|A\| = |y|$.
3. Suppose $\boldsymbol{f}$ maps a convex set E of $\mathbb{R}^n$ into $\mathbb{R}^m$, $\boldsymbol{f}$ is differentiable in E, and there is a real number M such that $\|\boldsymbol{f}'(x)\| \leq M$ for every $\boldsymbol{x} \in E$.
   Prove that $|\boldsymbol{f}(\boldsymbol{b}) - \boldsymbol{f}(\boldsymbol{a})| \leq M|\boldsymbol{b} - \boldsymbol{a}|$.
4. Prove that if the tangent vector of a parametrized curve is constant, the image of the curve is a straight line.
5. Give an example of a subset of $\mathbb{R}^3$ which is not a surface.
6. Verify whether $\sigma(u, v) = (u + u^2, v, v^2), u, v \in \mathbb{R}$ a regular surface patch or not.
7. Compute the first fundamental form of the surface $\sigma(u, v) = (u - v, u + v, u^2 + v^2)$.
8. Compute the second fundamental form of the elliptic paraboloid $\sigma(u, v) = (u, v, u^2 + v^2)$.

(8 × 1 = 8 weightage)

Part B: Answer any *two* questions from each unit. Each carries 2 weightage.

Unit 1

9. Let S is a metric space, if $a_{11} \dots a_{mn}$ are real continuous functions on S, and for each $p \in S$,A_p is the linear transformation of $\mathbb{R}^n$ into $\mathbb{R}^m$ whose matrix has entries $a_{ij}(p)$, prove that $\|A_p\| \leq \{\sum_{i,j} a_{ij}(p)^2\}^{\frac{1}{2}}$. Also prove that the mapping $p \to A_p$ is a continuous mapping of S into $L(\mathbb{R}^n, \mathbb{R}^m)$.
10. Suppose f maps an open set $E \subset \mathbb{R}^n$ into $\mathbb{R}^m$. If the partial derivatives $D_j f_i$ exist and are continuous on E for $1 \leq i \leq m, 1 \leq j \leq n$, then prove that $f \in \mathcal{C}'(E)$.
11. If $[A]$ and $[B]$ are n by n matrices, then show that $\det[B][A] = \det[B]\det[A]$.

(P.T.O.)

Unit 2

12. Show that the relation 'being a reparametrization of' is an equivalence relation on the set of parametrized curves in $\mathbb{R}^n$.

13. Prove that the curvature of a regular curve $\gamma(t)$ in $\mathbb{R}^3$ is $\kappa = \frac{\|\ddot{\gamma}\times\dot{\gamma}\|}{\|\dot{\gamma}\|^3}$. Also find the curvature of the circular helix.

14. If $f: S_1 \to S_2$ is a diffeomorphism between surfaces and $p \in S_1$, prove that the derivative $D_p f: T_p S_1 \to T_{f(p)} S_2$ is a linear map and is invertible.

Unit 3

15. Prove that the transition maps of a smooth surface are smooth.

16. Let $\sigma(u, v)$ be a surface patch with standard unit normal $\mathbf{N}(u, v)$. Then prove that $\mathbf{N}_u \cdot \sigma_u = -L, \mathbf{N}_u \cdot \sigma_v = \mathbf{N}_v \cdot \sigma_u = -M, \mathbf{N}_v \cdot \sigma_v = -N$.

17. Show that the Gaussian curvature of a ruled surface is negative or zero.

(6 × 2 = 12 weightage)

Part C: Answer any *two* questions. Each carries 5 weightage.

18. Prove the following.

    a) If $A \in L(\mathbb{R}^n, \mathbb{R}^m)$, then $\|\mathrm{A}\| < \infty$ and A is a uniformly continuous mapping of $\mathbb{R}^n$ into $\mathbb{R}^m$.

    b) If $\mathrm{A}, \mathrm{B} \in \mathrm{L}(\mathbb{R}^n, \mathbb{R}^m)$ and c is a scalar, then $\|\mathrm{A} + \mathrm{B}\| \leq \|\mathrm{A}\| + \|\mathrm{B}\|, \|\mathrm{cA}\| = |\mathrm{c}|\|\mathrm{A}\|$.

    c) If $\mathrm{A} \in \mathrm{L}(\mathbb{R}^n, \mathbb{R}^m)$, and $\mathrm{B} \in \mathrm{L}(\mathbb{R}^m, \mathbb{R}^\kappa)$, then $\|\mathrm{BA}\| \leq \|\mathrm{B}\|\|\mathrm{A}\|$.

19. State and prove Inverse function theorem.

20. a) Let $k: (\alpha, \beta) \to \mathbb{R}$ be any smooth function , prove that there is a unit speed curve $\gamma: (\alpha, \beta) \to \mathbb{R}^2$ whose signed curvature is k.If $\bar{\gamma}: (\alpha, \beta) \to \mathbb{R}^2$ is any unit speed curve whose signed curvature is k, how does γ and $\bar{\gamma}$ are related?

    b) Prove that any regular curve whose curvature is a positive constant is part of a circle.

21. a) Let σ be a surface patch of an oriented surface S with first and second fundamental forms $Edu^2 + 2Fdudv + Gdv^2$ and $Ldu^2 + 2Mdudv + Ndv^2$ respectively.

    Let $\mathcal{F}_I = \begin{pmatrix} E & F \\ F & G \end{pmatrix}$ and $\mathcal{F}_{II} = \begin{pmatrix} L & M \\ M & N \end{pmatrix}$ then prove that the matrix $\mathcal{W}_{p,S}$ of Weingarten map with respect to the basis of $\{\sigma_u, \sigma_v\}$ of $T_p S$ is $\mathcal{F}_I^{-1}\mathcal{F}_{II}$.

    b) Prove that the mean curvature $H = \frac{LG-2MF+NE}{2(EG-F^2)}$ and Gaussian curvature $K = \frac{LN-M^2}{EG-F^2}$.

(2 × 5 = 10 weightage)