D3AMT2303 **(2 Pages)** **Name............................**
Reg.No..........................

THIRD SEMESTER M. Sc. DEGREE EXAMINATION, NOVEMBER 2024
(Regular/Improvement/Supplementary)

MATHEMATICS
FMTH3C13- FUNCTIONAL ANALYSIS

Time: 3 Hours **Maximum Weightage: 30**

Part A: Answer *all* questions. Each carries 1 weightage.

1. Describe the sequence spaces $c_0, c,$ and ℓ_∞. Explain the inclusion relation among these spaces.
2. Define a norm on the space $C[a,b]$, and verify that it is indeed a norm.
3. Show that the sequence space ℓ_q is proper subspace of ℓ_p, where $1 \le q < p < \infty$.
4. Give an example of an inner product space which is not a Hilbert space. Justify it.
5. For any vectors x, y in an inner product space, prove that:
$\|x+y\|^2 + \|x-y\|^2 = 2(\|x\|^2 + \|y\|^2)$.
6. For any $x \in H$ and any orthonormal system $\{e_i\}_1^\infty$, prove that there exists a $y \in H$ such that $y = \sum_{i=1}^{\infty} \langle x, e_i \rangle e_i$.
7. For any x_1, x_2 in a normed space with $x_1 \neq x_2$, prove that there exists a bounded linear functional f such that $f(x_1) \neq f(x_2)$.
8. Show that the dual operator A^* of a bounded operator A is bounded and $\|A\| = \|A^*\|$.

(8 × 1 = 8 weightage)

Part B: Answer *any two* questions *from each unit*. Each carries 2 weightage.

Unit 1

9. State and prove Holder's inequality for scalar sequences.
10. If X is a Banach space and if E is a closed subspace of X, prove that the quotient space X / E is a Banach space.
11. Define the completion of a normed space and show that every normed space has a completion.

(P.T.O.)

Unit 2

12. State and prove Cauchy-Schwartz inequality in an inner product space.
13. Define orthogonal complement. If L is a closed subspace of a Hilbert space, then prove that $\left(L^{\perp}\right)^{\perp} = L.$
14. Show that a Hilbert space is separable if and only if there exists a complete orthonormal system.

Unit 3

15. For $1 < p < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$, prove that $\left(\ell_p\right)^* = \ell_q$.
16. For any bounded linear operator A on a Hilbert space H, prove that:

$$\|A\| = \sup\left\{\left|\langle Ax, y\rangle\right| \; : \; \|x\| \le 1, \|y\| \le 1\right\}.$$

17. Show that every bounded operator of finite rank is a compact operator.

(6 × 2 = 12 weightage)

Part C: Answer any *two* questions. Each carries 5 weightage.

18. (a) State and prove Minkowski's inequality for scalar sequences.

    (b) Show that the sequence space c_0 with sup norm is a Banach space.

19. (a) State and prove Riesz representation theorem.

    (b) Show that any two separable infinite dimensional Hilbert spaces are isometrically equivalent.

20. (a) If X is a normed space and if Y is a complete normed space, prove that the space $L(X \mapsto Y)$ is a Banach space.

    (b) Show that all ℓ_p spaces, $1 < p < \infty$ are reflexive.

21. (a) Define the three different notions of convergence in the space of bounded operators. Distinguish these using examples.

    (b) Let H be a Hilbert space and $A : H \to H$ be a linear operator. Prove that A is compact if and only if its adjoint A^* is compact.

(2 × 5 = 10 weightage)