D3AMT2302 **Name..............................**

Reg. No. ....

THIRD SEMESTER M.Sc. DEGREE EXAMINATION, NOVEMBER 2024
(Regular/Improvement/Supplementary)
MATHEMATICS
FMTH3C12: COMPLEX ANALYSIS

Time : 3 Hours **Maximum Weightage: 30**

Part A
Answer *all* questions. Each carries 1 weightage.

1. Discuss the $d(z, \infty)$ in the extended complex plane.
2. Define radius of convergence R and find R for the series $\Sigma_{n=0}^{\infty} z^{n!}$.
3. Find the image of $\{z : Rez < 0, |\ Imz\ | < \pi\}$ under the exponential function.
4. Solve $\int_\gamma \frac{e^{iz}dz}{z^2}, \gamma(t) = re^{it}, 0 \leq t \leq 2\pi$.
5. Let G be an open set which is star shaped. If γ_0 is the curve which is constantly equal to a, then prove that every closed rectifiable curve in G is homotopic to γ_0.
6. Prove that $n(\gamma, a) = -n(-\gamma, a)$ where γ is a closed rectifiable curve.
7. Determine the type of singularity of the function $f(z) = \frac{log(z+1)}{z^2}$.
8. State Casorati- Weierstrass Theorem.

(8 $\times$ 1 = 8 weightage)

Part B
Answer *any two* questions from *each unit.* Each carries 2 weightage.

Unit 1

9. Let f and g be analytic on G and Ω rspectively and $f(G) \subset \Omega$, then prove that $g \circ f$ is analytic and $(g \circ f)' = g'(f(z))f'(z) \quad \forall z \in G$.
10. If $G \subset C$ is open and connected and f is a branch of $logz$ on G, then prove that the totality of branches of $logz$ are functions $f(z) + 2\pi ki, k \in Z$.
11. Let G be either the whole plane C or some open disk. If $u : G \to R$ is a harmonic function, then prove that u has a harmonic conjugate.

Unit 2

12. Prove that if f is analytic in $B(a, r)$, then $f(z) = \Sigma_{n=0}^{\infty} a_n (z-a)^n$ for $| z - a |< R$ where $a_n = \frac{1}{n!} f^{(n)}(a)$ and this series has radius of convergence greater than or equal to R.
13. Let f be an entire function and suppose that there is a constant M, an $R > 0$ and an integer $n \geq 1$ such that $| f(z) |\leq M | z |^n$ for $| z |> R$. Show that f is a polynomial of degree $\leq n$.
14. Let G be simply connected and let $f : G \to C$ be an anlytic function such that $f(z) \neq 0$ for any z in G. Then prove that there is an analytic function $g : G \to C$ such that $f(z) = e^{g(z)}$. Prove that if $z_0 \in G$ and $e^{w_0} = f(z_0)$, we can choose g such that $g(z_0) = w_0$.

Unit 3

15. If G is a region with a in G and if f is analytic on $G - \{a\}$ with pole at $z = a$ then prove that there is a positive integer m and an anlytic function $g : G \to C$ such that $f(z) = \frac{g(z)}{(z-a)^m}$.
16. State and prove Residue Theorem.
17. If $| a |< 1$ and φ_a is a one-one map of $D = \{z :| z |< 1\}$onto itself, then prove that the inverse of φ_a is φ_{-a}. Also prove that φ_a maps ∂D onto ∂D, $\varphi_a(a) = 0$, $\varphi_a^{'}(0)$ $=1 - (| a |)^2$ and $\varphi_a^{'}(a)$ $=(1 - (| a |)^2)^{-1}$.

(6 $\times$ 2 = 12 weightage)

Part C

Answer *any two* questions. Each carries 5 weightage.

18. (a) Let (z_1, z_2, z_3, z_4) be four distinct points in C_∞. Prove that (z_1, z_2, z_3, z_4) is a real number if and only if all four points lie on a circle.

    (b) Prove that Mobius transformation takes circles on to circles.
19. (a) Prove that for a given power series $\Sigma_{n=0}^{\infty} a_n (z-a)^n$ if we define the number R, $0 \leq R \leq \infty$ by $\frac{1}{R} = limsup \mid a_n \mid^{1/n}$ then

    i. If $| z - a |< R$, the series converges absolutely.

    ii .$| z - a |> R$ the terms of the series become unbonded and so the series diverges.

    iii. If $0 < r < R$ then the series converges uniformly on $\{z :| z |\leq r$.

    (b) If $\Sigma a_n (z - a)^n$ is a power series with radius of convergence R then prove that $R = lim \mid \frac{a_n}{a_{n+1}} \mid$ if it exist.
20. State and prove the Laurent series development of an analytic function.
21. State and Prove Hadamad's Three Circles Theorem.

(2 $\times$ 5= 10 weightage)