D3AMT2301 (2 Pages) Name............................
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THIRD SEMESTER M. Sc. DEGREE EXAMINATION, NOVEMBER 2024
(Regular/Improvement/Supplementary)

MATHEMATICS
FMTH3C11- MULTIVARIABLE CALCULUS AND GEOMETRY

Time: 3 Hours **Maximum Weightage: 30**

Part A: Answer *all* questions. Each carries 1 weightage.

1. Let X, Y, Z be vector spaces. If $A \in L(X, Y)$ and $B \in L(Y, Z)$ then prove that the composition BA is a linear transformation from X to Z.
2. Show that if I is the identity operator on R^n, then $det[I] = det(e_1, e_2, \dots, e_n) = 1$.
3. Find the parametrization of the level curve $y^2 - x^2 = 1$.
4. Give an example to show that a level curve can have both regular and nonregular parametrizations.
5. Find the equation of the tangent plane of the surface patch $\sigma(u, v) = (u. v. u^2 - v^2)$ at the point (1,1,0).
6. Check whether the circular cone $f(x, y, z) = x^2 + y^2 - z^2$ is a smooth surface.
7. Calculate the first fundamental form of the plane $\sigma(u, v) = a + u\boldsymbol{p} + v\boldsymbol{q}$.
8. Define Weingarten map.

(8 × 1 = 8 weightage)

Part B: Answer any *two* questions from each unit. Each carries 2 weightage.

Unit 1

9. If Ω is the set of all invertible linear operators on R^n, then prove that the mapping $A \to A^{-1}$ is continuous on Ω.
10. Suppose f maps an open set $E \subset R^n \to R^m$ and f is differentiable at a point $x \in E$,then prove that the partial derivatives $D_j f_i(x)$ exist and $f'(x)e_j = \sum_{i=1}^{m} D_j f_i\,(x) u_i\,; 1 \leq j \leq n$, where $\{e_1, e_2, e_3, \dots, e_n\}$ and $\{u_1, u_2, u_3, \dots, u_m\}$ are the standard basis of R^n and R^m respectively.
11. Suppose f maps a convex open set $E \subset R^n \; into \; R^m$ and $f'(x) = 0$ for all $x \in E$, then prove that f is a constant.

(P.T.O.)

Unit 2

12. Prove that the unit speed parametrization of a regular closed curve is closed.
13. Let γ be a unit- speed curve in R^3 with constant curvature and zero tension. Prove that γ is parametrization of (part of) a circle.
14. Let $\sigma: U \to R^3$ be a patch of a surface S containing a point $p \in S$, and let (u, v) be coordinates in U. Prove that the tangent space to S at p is the vector subspace of R^3 spanned by the vectors σ_u and σ_v.

Unit 3

15. Compute the second fundamental form of the elliptic paraboloid $\sigma(u,, v) = (u, v, u^2 + v^2)$.
16. Prove that the Gaussian curvature of a ruled surface is negative or zero.
17. Find the principal curvatures of the unit cylinder $\sigma(u, v) = (\cos v, \sin v\,, u)$.

(6 × 2 = 12 weightage)

Part C: Answer any *two* questions. Each carries 5 weightage.

18. a) Let the distance between A and B in $L(R^n, R^m)$ is defined as $\|A - B\|$. Prove that $L(R^n, R^m)$ is a metric space.

    b) Prove that if X is a complete metric space and φ is a contraction from X into X, then φ has a fixed point in X.

19. a) Suppose f is a continuously differentiable mapping on an open set $E \subset R^n$ into R^n, $f'(a)$ is invertible for some $a \in E$ and $b = f(a)$.Then prove that there exist open sets U and V such that $a \in U$,$b \in V$, f is one to one on U and $f(U) = V$.

    b) Let $f: R^2 \to R^3$ be defined as $f(x, y) = (3x - y^2, 2x + y, xy + y^3)$ and $g: R^2 \to R^2$ be defined as $g(x, y) = (2ye^{2x}, xe^y)$. Show that there are neighbourhoods U of (0,1) and V of (2,0) such that g maps U bijectively onto V.

20. Let $\gamma: (\alpha, \beta) \to R^2$ be a unit speed curve, let $s_0 \in (\alpha, \beta)$ and let φ_0 be such that $\dot{\gamma}(s_0) = (\cos \varphi_0, \sin \varphi_0)$. Then prove that there is a unique smooth function $\varphi: (\alpha, \beta) \to R$ such that $\varphi(s_0) = \varphi_0$ and that the equation $\dot{\gamma}(s) = (\cos \varphi(s), \sin \varphi(s))$ holds for all $s \in (\alpha, \beta)$.
21. Let $\sigma: U \to R^3$ be a surface patch. Let $(u_0, v_0) \in U$, and let $\delta > 0$ be such that the closed disc $R_\delta = \{(u, v) \in R^2 / (u - u_0)^2 + (v - v_0)^2 \leq \delta^2\}$ with centre (u_0, v_0) and radius δ is contained in U. Then prove that $\lim_{\delta \to 0} \frac{\mathcal{A}_N(R_\delta)}{\mathcal{A}_\sigma(R_\delta)} = |K|$, where K is the Gaussian curvature of σ at (0, 0).

(2 × 5 = 10 weightage)