D3AMT2201 **(2 Pages)** **Name.............................**
Reg.No...........................

THIRD SEMESTER M. Sc. DEGREE EXAMINATION, NOVEMBER 2023
(Regular/Improvement/Supplementary)

MATHEMATICS
FMTH3C11- MULTIVARIABLE CALCULUS AND GEOMETRY

Time: 3 Hours **Maximum Weightage: 30**

Part A: Answer *all* questions. Each carries 1 weightage.

1. Let $B = \{x_1, x_2, x_3, \dots\dots, x_n\}$ be a basis for a vector space V of dimension n.Then prove that every element x of V has a unique representation of the form $x = \sum_{j=1}^{n} c_j x_j$.
2. If$A \in L(R^n, R^m)$, then show that A is a uniformly continuous mapping of R^n into R^m.
3. Calculate the tangent vector at each point of the curve $\gamma(t) = (e^t, t^2)$.
4. Calculate the curvature of the circle $\gamma(t) = (x_0 + R\cos\frac{t}{R}, y_0 + R\sin\frac{t}{R})$.
5. Check whether $\sigma(u, v)$ =(u,v,uv) is a regular surface patch.
6. Show that the unit sphere S^2 is a smooth surface in R^3.
7. Calculate the mean curvature and Gaussian curvature of the surface whose principal curvatures are k_1 and k_2.
8. Define Weingarten map.

(8 × 1 = 8 weightage)

Part B: Answer any *two* questions from each unit. Each carries 2 weightage

Unit 1

9. Prove that a linear operator A on a finite dimensional vector space X is one to one if and only if the range of A is all of X.
10. Prove that the set of all invertible operators on R^n is an open subset of R^n.
11. Suppose $A \in L(R^{n+m}, R^n)$ and A_x is invertible. Prove that corresponding to every $k \in R^m$, there exists a unique $h \in R^n$ such that $A(h, k) = 0$.

(P.T.O.)

Unit 2

12. Let $\sigma: U \to R^3$ be a patch of a surface S containing a point $p \in S$, and let (u, v) be coordinates in U. Prove that the tangent space to S at p is the vector subspace of R^3 spanned by the vectors σ_u and σ_v.
13. Find the unit speed reparametrization of the logarithmic spiral $\gamma(t) = (e^{kt} \cos t, e^{kt} \sin t)$, where k is a non-zero constant.
14. Let $k: (\alpha, \beta) \to R$ be any smooth function, then prove that there is a unit speed curve $\gamma: (\alpha, \beta) \to R^2$ whose signed curvature is k.

Unit 3

15. Calculate the first fundamental form of the surface $\sigma(u,, v) = (u - v, u + v, u^2 + v^2)$.
16. Show that the normal curvature of any curve on a sphere of radius r is $\pm\frac{1}{r}$.
17. Let S be a connected surface of which every point is an umbilic. Prove that S is an open subset of a plane or a sphere.

(6 × 2 = 12 weightage)

Part C: Answer any *two* questions. Each carries 5 weightage.

18. a) Prove that if X is a complete metric space and φ is a contraction from X into X then φ has a fixed point in X.

    b) Suppose f is a continuously differentiable mapping on an open set $E \subset R^n$ into $R^n, f'(a)$ is invertible for some $a \in E$ and $b = f(a)$. Then prove that there exist open sets U and V such that $a \in U$, $b \in V$, f is one to one on U and $f(U) = V$.
19. Show that a smooth map $f: S \to \tilde{S}$ is a local diffeomorphism if and only if, for all $p \in S$, the linear map $D_P f: T_p S \to T_{f(P)} \tilde{S}$ is invertible, where S and $\tilde{S}$ are surfaces.
20. Suppose f maps an open set $E \subset R^n$ into R^m.Then prove that $f \in \mathcal{C}'(E)$ if and only if the partial derivatives $D_j f_i$ exist and are continuous on E for $1 \leq i \leq m, 1 \leq j \leq n$.
21. Find the second fundamental form and the Gaussian curvature of the surface of revolution $\sigma(u, v) = (f(u) \cos v, f(u) \sin v, g(u))$; $f(u) > 0$ for all values of u and $\dot{f}^2 + \dot{g}^2 = 1$.

(2 × 5 = 10 weightage)