D3AMT2103 **(2 Pages)** **Name.............................**
Reg.No..........................

THIRD SEMESTER M. Sc. DEGREE EXAMINATION, NOVEMBER 2022
(Regular/Improvement/Supplementary)

MATHEMATICS
FMTH3C13- FUNCTIONAL ANALYSIS

Time: 3 Hours **Maximum Weightage: 30**

Part A: Answer *all* questions. Each carries 1 weightage.

1. Define a norm on λ_∞. Verify the conditions.
2. What do you mean by a Banach space? Give an example of a normed space which is not a Banach space.
3. If (x_n) is a Cauchy sequence in a normed space, then show that the sequence $(\|x_n\|)$ is always convergent.
4. State and prove Parseval's Identity.
5. If L is a closed subspace of a Hilbert space, then prove that $\left(L^\perp\right)^\perp = L.$
6. If a linear functional is continuous at $x = 0,$ then show that it is continuous at any $x.$
7. Show that the dual operator A^* of a bounded operator A is bounded and $\|A\| = \|A^*\|$.
8. Define three different notions of convergence in the space of bounded operators.

(8 × 1 = 8 weightage)

Part B: Answer *any two* questions *from each unit*. Each carries 2 weightage.

Unit 1

9. State and prove Holder's inequality for scalar sequences.
10. Show that the sequence space λ_p is proper subspace of λ_q, where $1 \le p < q < \infty.$
11. Define the completion of a normed space and show that every normed space has a completion.

(P.T.O.)

Unit 2

12. Define inner product. Show that an inner product is a continuous function with respect to both variables.

13. State and prove Riesz representation theorem.

14. Show that a Hilbert space is separable if and only if there exists a complete orthonormal system.

Unit 3

15. For $1 < p < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$, prove that $(\lambda_p)^* = \lambda_q$.

16. Define reflexive spaces. Show that every finite dimensional normed space is a reflexive space.

17. Define compact operator. Show that every bounded operator of finite rank is a compact operator.

(6 × 2 = 12 weightage)

Part C: Answer *any two* questions. Each carries 5 weightage.

18. a) Show that the sequence space c_0 with supremum norm is a Banach space.

    b) Define Quotient space. If X is a Banach space and if E is a closed subspace of X, show that the quotient space X / E is a Banach space.

19. a) Describe the Gram-Schmidt orthogonalization process.

    b) If L is a closed subspace of a Hilbert space H, then prove that $H = L \oplus L^{\perp}$.

20. a) Let H be a Hilbert space and $A : H \to H$ be a linear operator. Prove that A is compact if and only if its adjoint $A*$ is compact.

    b) Define relatively compact sets. Show that M is relatively compact if and only if for every $\varepsilon > 0$, there exists a finite $\varepsilon -$net in M.

21. a) If X is a normed space and if Y is a complete normed space, prove that the space

    $L(X \ \alpha \ Y)$ is a Banach space.

    b) For any bounded linear operator A on a Hilbert space H, prove that

    $$\|A\| = \sup \{ |\langle Ax, y \rangle| \ : \ \|x\| \leq 1, \|y\| \leq 1 \}.$$

(2 × 5 = 10 weightage)