D3AMT2101 **(2 Pages)** **Name.............................**
Reg.No...........................

THIRD SEMESTER M. Sc. DEGREE EXAMINATION, NOVEMBER 2022
(Regular/Improvement/Supplementary)

MATHEMATICS
FMTH3C11- MULTIVARIABLE CALCULUS AND GEOMETRY

Time: 3 Hours **Maximum Weightage: 30**

Part A: Answer *all* questions. Each carries 1weightage.

1. If $A \in L(R^n, R^m)$ and if $x \in R^n$, then prove that $A'(x) = A$.
2. Compute the directional derivative of $f: R^2 \to R$ defined by $f(x, y) = xy$, at (u, v) in the direction of (a, b).
3. Find the tangent vector at $t = \frac{2\pi}{3}$ of the parametrized curve

   $\gamma(t) = ((1 + 2\cos t)\cos t, (1 + 2\cos t)\sin t), t \in R$.
4. Check whether $\gamma(t) = (t, \cosh t)$ for $t \in R$ is a regular curve in R^2.
5. Show that the plane $\sigma(u. v) = a + \boldsymbol{p}u + \boldsymbol{q}v$ is a smooth surface in R^3.
6. Find the unit normal at any point on the surface represented by $\sigma(u, v)$ = (u, v,$u^2 + v^2$).
7. Define Gauss map.
8. Show that the Weingarten map W of a surface satisfies the quadratic equation

   $W^2 - 2HW + K = 0$ where H is the mean curvature and K is the Gaussian curvature of the surface.

(8 × 1 = 8 weightage)

Part B: Answer any *two* questions from each unit. Each carries 2 weightage.

Unit 1

9. Let r be a positive integer. If a vector space X is spanned by a set of r vectors, then prove that dimension of X is less than or equal to r.
10. Let $f: R^n \to R^m$ be a function. Prove that the derivative of f (if it exists) is unique.
11. Let X be a complete metric space and φ be a contraction of X into X. Prove that φ has a fixed point in X.

(P.T.O.)

Unit 2

12. Define closed curve. Show that the unit speed reparametrization of a regular closed curve is always closed.

13. Compute the curvature of the circular helix $\gamma(\theta) = (a\cos\theta, a\sin\theta, b\theta); \theta \in R$.

14. Let $\sigma: U \to R^3$ be a patch of a surface S containing a point $p \in S$, and let (u, v) be coordinates in U. Prove that the tangent space of S at p is the vector subspace of R^3 spanned by the vectors σ_u and σ_v.

Unit 3

15. Compute the first fundamental form of the surface represented by

    $$\sigma(u, v) = (u - v, u + v, u^2 + v^2).$$

16. Prove that the second fundamental form is a symmetric bilinear form.

17. Prove that the Gaussian curvature of a ruled surface is negative or zero.

(6 × 2 = 12 weightage)

Part C: Answer any *two* questions. Each carries 5 weightage.

18. Suppose f maps an open set $E \subset R^n$ into R^m. Then prove that $f \in \mathcal{C}'(E)$ if and only if the partial derivatives $D_j f_i$ exist and are continuous on E for $1 \leq i \leq m, 1 \leq j \leq m$.

19. State and prove the Inverse Function Theorem.

20. Let $\gamma: (\alpha, \beta) \to R^2$ be a unit speed curve, let $s_0 \in (\alpha, \beta)$ and let φ_0 be such that

    $\dot{\gamma}(s_0) = (\cos\varphi_0, \sin\varphi_0)$, then prove that there is a unique smooth function $\varphi: (\alpha, \beta) \to R$ such that $\varphi(s_0) = \varphi_0$ and that the equation $\dot{\gamma}(s) = (\cos \varphi(s), \sin \varphi(s)$ holds for all $s \in (\alpha, \beta)$.

21. a) Find the principal curvatures of the unit cylinder represented by the surface patch

    $$\sigma(u, v) = (\cos v, \sin v, u).$$

    b) Let S be a connected surface of which every point is an umbilic. Prove that S is an open subset of a plane or a sphere.

(2 × 5 = 10 weightage)