D3AMT2005 **(2 Pages)** **Name:.............................**

Reg.No:...........................

THIRD SEMESTER M.Sc DEGREE EXAMINATION NOVEMBER 2021
(Regular/Improvement/Supplementary)
MATHEMATICS
FMTH3E03: MEASURE AND INTEGRATION

Time: 3 Hours **Maximum Weightage: 30**

Part A: All questions can be answered. Each carries one weightage. (Ceiling 6 weightage)

1. Show that an extended real-valued function defined on a nonempty set X can be written as the difference of two nonnegative functions.

2. Consider the statement: *Let* μ *be a positive measure on a* $\sigma-$*algebra* $\mathcal{M}$ *and* $A_1 \supseteq A_2 \supseteq A_3 \supseteq \ldots$ *be such that* $A_i \in \mathcal{M}$ *for each* i. *If* $\mu(A_1) < \infty$, *then* $\mu(A_n) \to \mu(A)$, *where* $A = \cap_{n=1}^{\infty} A_n$. Show by example that the assumption, $\mu(A_1) < \infty$ cannot be dropped.

3. State True or False and justify your claim: If $(X, \mathcal{M}, \mu)$ is a measure space and $f : X \to [-\infty, \infty]$ is measurable, then
$$\int_X f d\mu = 0 \text{ if and only if } f = 0 \text{ almost everywhere.}$$

4. Define Lebesgue measurable subsets in $\mathbb{R}^k$ and Lebesgue measure on $\mathbb{R}^k$.

5. Show that the Lebesgue measure of a subspace Y of $\mathbb{R}^k$ is 0 if the dimension of Y is strictly less than k.

6. Show that if $\lambda_1, \lambda_2, \mu$ are measures on a $\sigma-$algebra $\mathcal{M}$ and μ is a positive measure, such that $\lambda_1 \perp \mu$ and $\lambda_2 \perp \mu$, then $\lambda_1 + \lambda_2 \perp \mu$.

7. State True or False and justify your claim: If $\mathscr{S}$ and $\mathscr{T}$ are $\sigma-$algebras on X and Y respectively and $E \in \mathscr{S} \times \mathscr{T}$, then the $x-$sections $E_x \in \mathscr{T}$ and $y-$sections $E^y \in \mathscr{S}$ for every $x \in X, y \in Y$.

8. Show by example that the product of two complete measure spaces need not be a complete measure space.

Part B: All questions can be answered. Each carries two weightage. (Ceiling of 12 weightage)

9. Let $(X, \mathcal{M}, \mu)$ be a measure space and $f : X \to \mathbb{C}$ be a complex-valued measurable function. Then explain rigorously the meaning of $\int_X f d\mu$.

10. State Riesz Representation Theorem for positive linear functionals on $C_c(X)$ where X is a locally compact Hausdorff space. Prove the uniqueness part.

11. (a) Show that if (f_n) is a sequence of nonnegative measurable functions then

$$\int_X (lim\,inf\, f_n) d\mu \leqslant lim\,inf \int_X f_n d\mu.$$

(b) Show by example that strict inequality can happen in the above case.

12. Show that there exists a subset of $[0, 1]$ that is not Lebesgue measurable.

13. Show that the Lebesgue measure of countable subset of $\mathbb{R}$ is 0. Is the converse true? Justify your claim.

14. Using Radon-Nikodym Theorem, establish the polar representation of a complex measure μ.

15. Show by example that a function f on the product space $X \times Y$ need not be $\mathscr{I} \times \mathscr{T}-$measurable even when f^x is $\mathscr{T}$-measurable and f_y is $\mathscr{I}-$measurable for every $x \in X, y \in Y$.

16. Show that the identity,

$$\int d\mu(x) \int f(x, y) d\lambda(y) = \int d\lambda(y) d\mu(x) \int f(x, y)$$

need not hold even if the two iterated integrals exist and finite.

17. Let m_l denotes the Lebesgue measure on $\mathbb{R}^l$ for $l \in \mathbb{N}$. Is the product measure $m_r \times m_s$ complete?

Part C: All questions can be answered. Each carries six weightage. (Ceiling 12 weightage)

18. (a) Introduce the vector space $L^1(\mu)$ and show that the map $f \mapsto \int f d\mu$ is a positive linear functional on $L^1(\mu)$.

(b) State and prove Lebesgue Dominated Convergence Theorem.

19. (a) Let X be a compact metric space. Show that there exist infinitely many non-constant bounded continuous functions on X.

(b) Give an example of a measure space $(X, \mathcal{M}, \mu)$ such that $\int_X f d\mu$ becomes an infinite sum for complex-valued measurable functions f defined on X.

20. Suppose f is a complex measurable function on X, $\mu(A) < \infty, f(x) = 0$ if $x \notin A$ and $\epsilon > 0$. Show that there exists a continuous function g on X with compact support such that

$$\mu(\{x : f(x) \neq g(x)\}) < \epsilon \quad \text{and} \quad \sup_{x \in X} |g(x)| \leqslant \sup_{x \in X} |f(x)|.$$

21. (a) State Fubini's theorem.

(b) Show by example that $\sigma-$finiteness assumption cannot be dropped.

(c) Show by example that integrability assumption on f cannot be dropped.