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THIRD SEMESTER M. Sc. DEGREE EXAMINATION, NOVEMBER 2021
(Regular/Improvement/Supplementary)

MATHEMATICS
FMTH3C13- FUNCTIONAL ANALYSIS

Time: 3 Hours **Maximum Weightage: 30**

Part A: All questions can be answered. Each carries one weightage (Ceiling 6 weightage).

1. Show that the linear space $C[a,b]$ of continuous functions is infinite dimensional.
2. Define linear independence relative to a subspace. Show that a set of vectors in a linear space is linearly independent if and only if it is linearly independent relative to the zero subspace.
3. Show that the sequence space c_0 with sup norm is a Banach space.
4. State and prove Cauchy-Schwartz inequality in an inner product space. .
5. Give example of an inner product space which is not a Hilbert space. Justify it.
6. Show that $\left\{\frac{1}{\sqrt{2\pi}}e^{\text{int}}\right\}_{n=-\infty}^{\infty}$ is an orthonormal system in the Hilbert space $L_2[a,b]$.
7. If L is a closed subspace of a normed space, then prove that $(\mathbf{L}^{\perp})^{\perp} = \mathbf{L}$.
8. Define strong convergence. Show that strong convergence is weaker than norm convergence.

Part B: All questions can be answered. Each carries two weightage (Ceiling 12 weightage).

9. State and prove Holder's inequality for functions.
10. Define quotient norm and verify that it is indeed a norm.
11. Define Banach spaces. Show that $C[a,b]$ with sup norm is a Banach space.
12. Define the inner product on ℓ_2 and verify that it is a Hilbert space.
13. Define complete system and give one example. If $\{f_i\}$ is a complete system in a Hilbert space H and $x \perp f_i$ for all i, then prove that $x = 0$.

(P.T.O.)

14. State and prove Riesz representation theorem.

15. Show that ℓ_1 is the dual space of c_0.

16. If X is a normed space and if Y is a complete normed space, prove that the space $L(X \mapsto Y)$ is a Banach space.

17. Let H be a Hilbert space and $A : H \to H$ be a linear operator. Prove that A is compact if and only if its adjoint A^* is compact.

Part C: All questions can be answered. Each carries six weightage (Ceiling 12 weightage).

18. (a) Define the completion of a normed space and show that every normed space has a completion.

    (b) Prove that the completion of a normed space is unique up to isometry.

19. (a) Prove that a Hilbert space H is separable if and only if there exists a complete orthonormal system $\{e_i\}_{i\geq 1}$ in H.

    (b) Show that every separable Hilbert space has an orthonormal basis.

20. (a) Show that all ℓ_p spaces, $1 < p < \infty$ are reflexive.

    (b) For any bounded linear operator A on a Hilbert space H, prove that

$$\|A\| = \sup \left\{ \left| \langle Ax, y \rangle \right| \; : \; \|x\| \leq 1, \|y\| \leq 1 \right\}$$

21. (a) Define compact operator. Show that the space $K(X \mapsto Y)$ is a closed subspace of $L(X \mapsto Y)$.

    (b) Show that every bounded operator of finite rank is a compact operator.