D3AMT2001 (2 Pages) Name............................
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THIRD SEMESTER M. Sc. DEGREE EXAMINATION, NOVEMBER 2021
(Regular/Improvement/Supplementary)

MATHEMATICS
FMTH3C11- MULTIVARIABLE CALCULUS AND GEOMETRY

Time: 3 Hours **Maximum Weightage: 30**

Part A: All questions can be answered. Each carries one weightage (Ceiling 6 weightage).

1. Let A be a linear transformation from the vector space R^n to R^m. Prove that $\|A\| < \infty$.
2. Compute the directional derivative of $f: R^2 \to R$ defined by $f(x, y) = xy$, at (u, v) in the direction of (a, b).
3. Illustrate using an example that the parametrization of a given level curve need not be unique.
4. Compute the curvature of the curve $\gamma(t) = (\frac{4}{5}\cos t, 1 - \sin t, \frac{-3}{5}\cos t)$.
5. Check whether $\sigma(u, v) = (u, v, uv)$ is a regular surface patch.
6. Compute the first fundamental form of the plane $\sigma(u. v) = a + \boldsymbol{p}u + \boldsymbol{q}v$.
7. Define Weingarten map.
8. If k_1 and k_2 are the principal curvatures of a surface S, then show that the mean curvature of the surface is $\frac{k_1+k_2}{2}$.

Part B: All questions can be answered. Each carries two weightage (Ceiling 12 weightage).

9. Let X be an n-dimensional vector space. If $1 \le r \le n$ and $\{y_1, y_2, \ldots\ldots, y_r\}$ is an independent set in X, then prove that X has a basis containing $\{y_1, y_2, \ldots\ldots, y_r\}$.
10. State and prove the Contraction Principle.
11. Prove that a linear operator A on R^n is invertible if and only if det [A] $\neq 0$.
12. Calculate the arc length of a logarithmic spiral $\gamma(t) = (e^{kt}\cos t, e^{kt} sint)$ starting at the point $\gamma(0) = (1,0)$, where k is a non-zero constant.
13. Let γ be a unit-speed curve in R^3 with constant curvature and zero tension. Prove that γ is a parametrization of a circle or part of a circle.
14. Define an orientable surface in R^3. Prove that Mobius band is not an orientable surface.
15. Compute the second fundamental form of the elliptic paraboloid $\sigma(u, v) = (u, v, u^2 + v^2)$.

(P.T.O.)

16. Prove that the Weingarten map is self-adjoint.

17. Find the principal curvatures of the unit cylinder represented by the surface patch $\sigma(u,v) = (\cos v, \sin v, u)$.

Part C: All questions can be answered. Each carries six weightage (Ceiling 12 weightage).

18. a) Prove that the set of all invertible linear operators on R^n is an open subset of $L(R^n)$.

    b) State and prove the chain rule for differentiation of functions of several variables.

19. State and prove the Inverse Function theorem.

20. Let $\gamma: (\alpha,\beta) \to R^2$ be a unit speed curve, let $s_0 \in (\alpha,\beta)$ and let φ_0 be such that $\dot{\gamma}(s_0) = (\cos\varphi_0, \sin\varphi_0)$. Then prove that there is a unique smooth function $\varphi: (\alpha,\beta) \to R$ such that $\varphi(s_0) = \varphi_0$ and that the equation $\dot{\gamma}(s) = (\cos \varphi(s), \sin \varphi(s)$ holds for all $s \in (\alpha,\beta)$.

21. a) Let $\sigma(u,v)$ be a surface patch with first and second fundamental forms $Edu^2 + 2Fdudv + Fdv^2$ and $Ldu^2 + 2Mdudv + Ndv^2$ respectively. Then prove that the Gaussian curvature K and the mean curvature H are $K = \frac{LG-2MF+NE}{2(EG-F^2)}$ and $H = \frac{LN-M^2}{EG-F^2}$ respectively.

    b) Compute the mean curvature and Gaussian curvatures of the surface of revolution $\sigma(u,v) = (f(u)\cos v, f(u)\sin v, g(u))$ where $f > 0$ and $\dot{f}^2 + \dot{g}^2 = 1$ everywhere (a dot denoting $\frac{d}{du}$).