D3AMT1905 **(2 Pages)** **Name............................**

Reg.No..........................

THIRD SEMESTER M.Sc. DEGREE EXAMINATION, NOVEMBER 2020
MATHEMATICS
FMTH3E03-MEASURE AND INTEGRATION

Time: 3 Hours Maximum Weightage: 30

Part A: Answer all questions. Each carries 1 weightage:

1. State True or False and justify your claim: If X is a measurable space and $E \subseteq X$, then the characteristic function $\chi_E : X \to \mathbb{R}$ defined by

$$\chi_E(x) = \begin{cases} 1 & \text{if} \quad x \in E \\ 0 & \text{if} \quad x \notin E \end{cases} \quad \text{is always measurable.}$$

2. State True or False and justify your claim: Let μ be a positive measure on a $\sigma-$algebra $\mathcal{M}$ and $A_1 \supseteq A_2 \supseteq A_3 \supseteq \ldots$ be such that $A_i \in \mathcal{M}$ for each i. Then

$$\mu(A_n) \to \mu(A), \text{ where } A = \cap_{n=1}^{\infty} A_n.$$

3. State True or False and justify your claim: The space $C_c(X)$ of all continuous functions with compact support is nontrivial if X is a locally compact Hausdorff space.
4. Define Lebesgue measurable subsets in $\mathbb{R}^k$ and Lebesgue measure on $\mathbb{R}^k$.
5. Show that if $\lambda_1, \lambda_2, \mu$ are measures on a $\sigma-$algebra $\mathcal{M}$ and μ is a positive measure, such that $\lambda_1 \perp \mu$ and $\lambda_2 \perp \mu$, then $\lambda_1 + \lambda_2 \perp \mu$.
6. State True or False and justify your claim: If μ is a $\sigma-$finite measure on a $\sigma-$algebra $\mathcal{M}$ in a set X, then every $w \in L^1(\mu)$ has the property that $w(x) = 1$ for some $x \in X$.
7. State True or False and justify your claim: If $\mathscr{I}$ and $\mathscr{T}$ are $\sigma-$algebras on X and Y respectively and $E \in \mathscr{I} \times \mathscr{T}$, then the $x-$sections $E_x \in \mathscr{T}$ and $y-$sections $E^y \in \mathscr{I}$ for every $x \in X, y \in Y$.
8. State True or False and justify your claim: If $(X, \mathscr{I}, \mu)$ and $(Y, \mathscr{T}, \lambda)$ are complete measure spaces, then $(X \times Y, \mathscr{I} \times \mathscr{T}, \mu \times \lambda)$ is also a complete measure space.

($8 \times 1 = 8$ **weightage)**

Part B: Answer any two questions from each unit. Each carries 2 weightage

Unit 1

9. Let $(X, \mathcal{M}, \mu)$ be a measure space and $f : X \to \mathbb{C}$ be a complex-valued measurable function. Then what you mean by $\int_X f d\mu$. Give all the details.
10. State Urysohn's Lemma. Give a simple proof in the case of a metric space.
11. (a) Does there exist a sequence of nonnegative measurable functions f_n such that

$$\int_X (lim\,inf\, f_n) d\mu < lim\,inf \int_X f_n d\mu.$$

(b) Does there exist a sequence of nonnegative measurable functions f_n such that

$$\int_X (lim\,inf\,f_n)d\mu > lim\,inf \int_X f_n d\mu.$$

Unit 2

12. Show that there exists a subset of $\mathbb{R}$ that is not Lebesgue measurable.

13. Show that there exists an uncountable subset of $\mathbb{R}$ with Lebesgue measure 0.

14. Using Radon-Nikodym Theorem, establish the polar representation of a complex measure μ.

Unit 3

15. Show that there exists a function f on the product space $X \times Y$ such that f^x is $\mathscr{T}$-measurable and f_y is $\mathscr{I}-$measurable for every $x \in X, y \in Y$, but f is not $\mathscr{I} \times \mathscr{T}-$measurable.

16. Show that the identity,

$$\int d\mu(x) \int f(x,y)d\lambda(y) = \int d\lambda(y)d\mu(x) \int f(x,y)$$

need not hold even if the two iterated integrals exist and finite.

17. If m_l denotes the Lebesgue measure on $\mathbb{R}^l$ for $l \in \mathbb{N}$ and if $k = r + s$, then show that m_k is the completion of $m_r \times m_s$.

($6 \times 2 = 12$ **weightage)**

Part C: Answer any two questions. Each carries 5 weightage

18. (a) Introduce $L^1(\mu)$ and show that it is a vector space. Give an example for a positive linear functional on $L^1(\mu)$, using the integration. Give all the details.

    (b) State and prove Lebesgue Dominated Convergence Theorem.

19. (a) State Riesz Representation Theorem for positive linear functionals on $C_c(X)$ where X is a locally compact Hausdorff space. Prove the uniqueness part.

    (b) Introduce a measure space $(X, \mathcal{M}, \mu)$ such that $\int_X f d\mu$ becomes an infinite sum for complex-valued measurable functions f defined on X.

20. Suppose f is a complex measurable function on X, $\mu(A) < \infty, f(x) = 0$ if $x \notin A$ and $\epsilon > 0$. Show that there exists a continuous function g on X with compact support such that

$$\mu(\{x : f(x) \neq g(x)\}) < \epsilon \quad \text{and} \quad \sup_{x\in X} |g(x)| \leqslant \sup_{x\in X} |f(x)|.$$

21. State and prove Fubini's theorem. Show that $\sigma-$finiteness assumption cannot be dropped.

($2 \times 5 = 10$ **weightage)**