**D3AMT1901** (2 PAGES) Reg. No. ···············

Name:··················

# THIRD SEMESTER M.Sc. DEGREE EXAMINATION, NOVEMBER 2020

## MATHEMATICS

## FMTH3C11 - MULTIVARIABLE CALCULUS AND GEOMETRY

**Time: 3 Hours** **Maximum Weightage: 30**

**Part A: Answer *all* questions. Each carries 1 weightage.**

1. Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be defined by $f(\mathbf{x}) = \mathbf{x}$, for every $\mathbf{x}$. Show that $f^{'}(\mathbf{x}) = I$, the identity operator in $L(\mathbb{R}^n)$.
2. Define a contraction map and give an example.
3. State inverse function theorem.
4. Is $\gamma(t) = (t^2, t^4)$, a parametrisation of the parabola $y = x^2$? Justify your claim.
5. Prove that the curvature of a straight line is zero.
6. Calculate the arc length of the logarithmic spiral $\gamma(t) = (e^t \cos t, e^t \sin t)$, starting at the point $(1, 0)$.
7. Define Weingarten map.
8. Find the first fundamental form of the surface $\sigma(u, v) = (u - v, u + v, u^2 + v^2)$.

**(8 × 1 = 8 Weightage)**

**Part B: Answer any *two* questions from each unit. Each carries 2 weightage.**

### Unit 1

9. If $f(x,y) = \begin{cases} \frac{xy}{x^2+y^2} & \text{if} \quad (x,y) \neq (0,0) \\ 0 & \text{if} \quad (x,y) = (0,0) \end{cases}$, prove that $D_1 f(x,y)$ and $D_2 f(x,y)$ exist at every point of $\mathbb{R}^2$, but $f$ is not continuous at $(0,0)$.
10. Prove that a linear operator $A$ on a finite dimensional vector space $X$ is one-to-one if and only if the range of $A$ is $X$.
11. Suppose that $f$ maps a convex open set $E \subseteq \mathbb{R}^n$ into $\mathbb{R}^m$ and $f$ is differentiable on $E$. If $f^{'}(\mathbf{x}) = 0$ for every $\mathbf{x} \in E$, then prove that $f$ is a constant function.

## Unit 2

12. Show that a parametrised curve has a unit speed re-parametrisation if and only if it is regular.

13. Find the torsion of a circular helix,$\gamma(\theta) = (a\cos\theta, a\sin\theta, b\theta); \theta \in \mathbb{R}$ and $a, b$ constants.

14. Let $f : S_1 \to S_2$ be a smooth map between surfaces and $p \in S_1$. Prove that the derivative $D_pf : T_pS_1 \to T_{f(p)}S_2$ is a linear map.

## Unit 3

15. With usual notations, prove that the mean curvature is

$$H = \frac{LG - 2MF + NE}{2(EG - F^2)}$$

16. If $k_1$ and $k_2$ are principal curvatures of a surface, then prove that the mean and Gaussian curvatures are given by $H = \frac{1}{2}(k_1 + k_2)$.

17. Prove that a curve on surface is a geodesic if and only if its geodesic curvature is zero everywhere.

**(6 × 2 = 12 Weightage)**

**Part C: Answer any *two* questions. Each carries 5 weightage.**

18. Let $E \subseteq \mathbb{R}^n$ be an open set and the map $f : E \to \mathbb{R}^k$ be differentiable at $\mathbf{x}_0 \in E$. If $g$ maps an open set containing $f(E)$ into $\mathbb{R}^m$ and $g$ is differentiable at $f(\mathbf{x}_0)$, then prove that the map $F : E \to \mathbb{R}^m$ defined by $F(\mathbf{x}) = g(f(\mathbf{x}))$ is differentiable at $\mathbf{x}_0$ and $F^{'}(\mathbf{x}_0) = g^{'}(f(\mathbf{x}_0))f^{'}(\mathbf{x}_0)$.

19. Let $\gamma$ be a unit speed curve in $\mathbb{R}^3$ with constant curvature and zero torsion. Prove that $\gamma$ is part of a circle.

20. Define diffeomorphism and local diffeomorphism. Prove that every diffeomorphism is a local diffeomorphism, but the converse is not true.

21. a) Prove that the normal curvature of any curve on a sphere of radius $r$ is $\pm\frac{1}{r}$.

    b) Calculate the Gaussian and mean curvature of the surface $\sigma(u, v) = (u+v, u-v, uv)$ at the point $(2, 0, 1)$.

**(2 × 5 = 10 Weightage)**