D2AMT2404 **Name................................**

Reg. No. ....

SECOND SEMESTER M.Sc. DEGREE EXAMINATION, APRIL 2025
(Regular/Improvement/Supplementary)
MATHEMATICS
FMTH2C09: ODE & CALCULUS OF VARIATIONS

Time : 3 Hours **Maximum Weightage: 30**

Part A: Answer *all* questions. Each carries 1 weightage.

1. Find the normal form of the ODE $y'' + P(x)y' + Q(x)y = 0$.
2. Locate and classify singular points of $(x^2 - 1)y'' + (2 - x)y' + (x + 1)y = 0$.
3. Give the Gauss's hypergeometric equation.
4. Determine the nature of the point $x = \infty$ for the equation $(1 - x^2)y'' - 2xy' + p(p + 1)y = 0$.
5. Give expressions for Legendre polynomials $P_0(x)$, $P_1(x)$, $P_2(x)$ and $P_3(x)$.
6. Determine whether $E(x, y) = -3x^2 + 2xy - y^2$ is positive definite, negative definite or neither.
7. Find the critical points of $\dfrac{d^2x}{dt^2} + \dfrac{dx}{dt} - (x^2 - 2x + 1) = 0$.
8. Define stable and asymptotically stable critical points.

(8 × 1 = 8 weightage)

Part B: Answer any *two* questions from *each unit.* Each carries 2 weightage.

Unit 1

9. State and prove Sturm comparison theorem.
10. Prove that the equation $4x^2y'' - 8x^2y' + (4x^2 + 1)y = 0$ has only one Frobenius series solution and find it.
11. Prove that $(1 + x)^p = F(-p, b, b, -x)$.

Unit 2

12. If $W(t)$ is the wronskian of the two solutions $\begin{cases} x = x_1(t) \\ y = y_1(t) \end{cases}$ and $\begin{cases} x = x_2(t) \\ y = y_2(t) \end{cases}$ of the homogeneous system $\begin{cases} \dfrac{dx}{dt} = a_1(t)x + b_1(t)y \\ \dfrac{dy}{dt} = a_2(t)x + b_2(t)y \end{cases}$ on $[a, b]$, then prove that $W(t)$ is either identically zero or nowhere zero on $[a, b]$.

(P.T.O.)

13. Show that $f(x,y) = x^2|y|$ satisfies a Lipschitz condition on the rectangle $|x| \leq 1$ and $|y| \leq 1$.

14. State and prove orthogonality properties of Legendre polynomials.

Unit 3

15. Find the general solution of the system $\begin{cases} \dfrac{dx}{dt} = x \\ \dfrac{dy}{dt} = -x + 2y. \end{cases}$

16. Describe the phase portrait of the system $\begin{cases} \dfrac{dx}{dt} = x \\ \dfrac{dy}{dt} = 0. \end{cases}$

17. Find the extremals for the integral $I = \int_{x_1}^{x_2} \dfrac{(\sqrt{1+(y')^2})}{y}$

(6 × 2 = 12 weightage)

Part C: Answer any *two* questions. Each carries 5 weightage.

18. Verify that the origin is a regular singular point and calculate two independent Frobenius series solutions of the equation
$2xy'' + (3-x)y' - y = 0.$

19. (a) State and prove Rodrigues' formula.

    (b) Prove that $\dfrac{d}{dx}[xJ_1(x)] = xJ_0(x)$.

20. (a)Determine the nature and stability properties of the critical point $(0,0)$ for

$$\begin{cases} \dfrac{dx}{dt} = -x - 2y \\ \dfrac{dy}{dt} = 4x - 5y. \end{cases}$$

    (b) Consider the autonomous system:

$$\begin{cases} \dfrac{dx}{dt} = -x \\ \dfrac{dy}{dt} = -2y. \end{cases}$$

    i) Find the general solution.
    ii) Find the differential equation of the paths.
    iii) Discuss the stability of the critical point.

21. (a) Find first three terms of the Legendre series of $f(x) = \begin{cases} 1 & \text{if } -1 \leq x < 0, \\ x & \text{if } 0 \leq x \leq 1 \end{cases}$

    (b) Solve $y' = y^2$, $y(0) = 1$ using Picard's method of successive approximations.

(2 × 5 = 10 weightage)