D2AMT2401 **Reg.No........................**

Name: ....

SECOND SEMESTER M.Sc. DEGREE EXAMINATION, APRIL 2025
(Regular/Improvement/Supplementary)
MATHEMATICS
FMTH2C06: GALOIS THEORY

Time : 3 Hours **Maximum Weightage: 30**

Part A

Answer *all* questions. Each carries 1 weightage.

1. Show that $\sqrt{\sqrt[3]{5}+i}$ is algebraic over $\mathbb{Q}$.
2. Which of the real numbers are consructible $\sqrt{2}$, $2^{\frac{1}{3}}$ and $5^{\frac{1}{3}}$.
3. Is any element of $\mathbb{Q}(\sqrt{2})$ a zero of x^3-2? Justify your answer.
4. Find all conjugates in $\mathbb{C}$ of $\sqrt{3-i}$ over $\mathbb{R}$.
5. Define perfect field and give one example.
6. State Conjugation Isomorphism Theorem.
7. Find the cyclotomic polynomial $\Phi_6(x)$ over $\mathbb{Q}$.
8. Define finite normal extesion of F.

(8 × 1 = 8 weightage)

Part B

Answer *any two* questions from *each unit.* Each carries 2 weightage.

Unit I

9. Let F be a field and let $f(x)$ be a nonconstant polynomial in F$[x]$. Prove that there exists an extension field E of F and an $\alpha \in$ E such that $f(\alpha)=0$.
10. Prove that $\mathbb{Q}(\sqrt{3}+\sqrt{7})=\mathbb{Q}(\sqrt{3},\sqrt{7})$.
11. Prove the existence of finite field $\mathbf{GF}(p^n)$ for every prime power p^n.

Unit II

12. Prove that $F_{\{\sigma_p\}} \simeq \mathbb{Z}_p$, Where σ_p is the Frobenius automorphism on F.
13. Find G$(\mathbb{Q}(\sqrt{2},\sqrt{3})/\mathbb{Q})$.
14. Let $E \leq \overline{F}$ is a splitting field over F. Prove that every irreducible polynomial in F$[x]$ having zero in E splits in E.

Unit III

15. Let K be a finite normal extension of a field F, and let F $\leq$ E $\leq$ K $\leq \overline{F}$. Prove that K is a finite normal extension of E and G(K/E) is the subgroup of G(K/F).

16. Let K be the splitting field of $x^4 + 1$ over $\mathbb{Q}$.

    (a) Show that $[K : \mathbb{Q}] = 4$.

    (b) Prove that G(K/$\mathbb{Q}$) is isomorphic to Kline 4-group.

17. Prove that the Galois group of the nth cyclotomic extesion of $\mathbb{Q}$ is isomorphic to the group consisting of the positive integers less than n and relatively prime to n under multiplication modulo n.

(6 $\times$ 2 = 12 weightage)

Part C

Answer *any two* questions. Each carries 5 weightage.

18. (a) Prove that $60°$ cannot be trisected.

    (b) Let p be a prime and let $n \in \mathbb{Z}^+$. If E and E' are fields of order p^n, then show that $E \simeq E'$.

19. (a) State and prove Isomorphic Extension Theorem.

    (b) Show that if [E : F] = 2, then E is a spliting field over F.

20. (a) Let E be a finite seperable extesion of a field F. Prove that there exists $\alpha \in$ E such that $E = F(\alpha)$.

    (b) Prove that a finite extension of a field of characteristic zero is a simple extension.

21. (a) Let F be a field of characteristic zero, and let $F \leq E \leq K \leq \overline{F}$, where E is a normal extesion of F and K is an extension of F by radicals. Prove that G(E/F) is a solvable group.

    (b) Is every polynimoal in F[x] of the form $ax^8 + bx^6 + cx^4 + dx^2 + e$, where $a \neq 0$, solvable by radicals over F, where F is of characteristic 0? Why or why not?

(2 $\times$ 5 = 10 weightage)