D2AMT2302 **Reg.No.........................**

Name: ....

SECOND SEMESTER M.Sc. DEGREE EXAMINATION, APRIL 2024
(Regular/Improvement/Supplementary)
MATHEMATICS
FMTH2C07: REAL ANALYSIS II

Time : 3 Hours **Maximum Weightage: 30**

Part A

Answer all questions. Each carries 1 weightage

1. Define Lebesgue Outer Measure. Show that every non-empty open set has positive measure.
2. Define Measurable function. Show that every continuous functions are measurable.
3. If f is integrable then show that f is finite valued a.e..
4. Let $f(x)$ be function defined on $[0, 2]$ as: f(x) =1 for x rational and $f(x) = -1$, if x is irrational. Then find $\int_0^2 f d\mu$.
5. Let f be a non negative bounded measurable function on a set of finite measure E. Then show that $\int_E f = 0$ if and only if $f = 0$ a.e. on E.
6. Let f be integrable over E and g be a bounded measurable function on E. Show that fg is integrable over E.
7. Let f be the Dirichlet function (characteristic function of the rationals in $[0, 1]$). Is f of bounded variation on $[0, 1]$?
8. If $f \in L^1(\mu)$ and $g \in L^\infty(\mu)$, then show that $fg \in L^1(\mu)$.

(8×1 = 8 weightage)

Part B

Answer any *two* questions from each unit. Each carries 2 weightage.

Unit I

9. Define $\sigma-$ algebra. Show that collection of all Lebesgue Measurable Set M is a $\sigma-$ algebra.
10. Show that there exists an uncountable set of measure zero.
11. State and prove Egoroff's Theorem.

(P.T.O.)

Unit II

12. State and prove Bounded Convergence Theorem.

13. Assume E has finite measure. Let the sequence of functions f_n be uniformly integrable over E. Show that if $f_n \to f$ pointwise a.e. on E, then f is integrable over E. Is the condition $m(E) < \infty$ necessary? Give reason your answer.

14. Define the converge in measure of a sequence f_n of functions. Show that if $f_n \to f$ in measure on E, then there is a subsequence f_{nk} that converges pointwise a.e. on E to f.

Unit III

15. Let f be a monotone function on the open interval (a, b). Then show that f is continuous except possibly at a countable number of points in (a, b),

16. Show that a function f is of bounded variation on the closed bounded interval $[a, b]$ if and only if it is the difference of two increasing functions on $[a, b]$.

17. State and prove Holders inequality. When does its equality occur?

(6×2 = 12 weightage)

Part C

Answer any *two* questions. Each carries 5 weightage.

18. .

a) State and prove The Simple Approximation Theorem.

b) Prove that if $E_1 \supseteq E_2 \supseteq \ldots$ and $m(E_i) < \infty$ for each i then show that $m(\lim E_i) = \lim m(E_i)$. Also show by an example that $m(E_i)$ must be finite for some i is necessary in this result.

19. a) Let f and g be bounded measurable functions on a set E of finite measure. Then for any α and β show that:
$\int_E (\alpha f + \beta g) = \alpha \int_E f + \beta \int_E g$.

b) State and prove Fatou's Lemma.

20. a) State and prove monotone convergence theorem.

b) State true or false. Monotone convergence theorem may not hold for decreasing sequence of functions. Justify your answer.

c) Show that the space $L^p(E)$ $\quad 1 \le p \le \infty$ form a normed vector space over real numbers.

21. a) State and prove Lebesgue's theorem.

b) Let E be a measurable set and $1 \le p \le \infty$. Then show that $L^P(E)$ is a Banach Space.

(2×5 = 10 weightage)