D2AMT2301 **Reg.No.........................**

Name: ....

SECOND SEMESTER M.Sc. DEGREE EXAMINATION, APRIL 2024
(Regular/Improvement/Supplementary)
MATHEMATICS
FMTH2C06: GALOIS THEORY

Time : 3 Hours **Maximum Weightage: 30**

Part A

Answer *all* questions. Each carries 1 weightage.

1. If $\alpha = \sqrt{\frac{1}{3} + \sqrt{7}}$, then determine deg$(\alpha, \mathbb{Q})$.
2. If α and β are constructible real numbers then prove that $\alpha + \beta$ is a constructible real number.
3. Does the polynomial $x^3 - 2$ split in $\mathbb{Q}(\sqrt{2})$? Justify your answer.
4. Find all conjugates in $\mathbb{C}$ of $\sqrt{1+\sqrt{2}}$ over $\mathbb{Q}(\sqrt{2})$.
5. Define Frobenius automorphism.
6. State Isomorphic Extension Theorem.
7. Let K be a finite normal extension of a field F, and let F $\leq$ E $\leq$ K $\leq \overline{F}$. Prove that K is a finite normal extension of E.
8. Find the cyclotomic polynomial $\Phi_8(x)$ over $\mathbb{Q}$.

(8 × 1 = 8 weightage)

Part B

Answer any *two* questions from each unit. Each carries 2 weightage.

Unit I

9. State and prove Fundamental Theorem of Algebra.
10. Prove that $\mathbb{Q}(2^{\frac{1}{2}}, 2^{\frac{1}{3}}) = \mathbb{Q}(2^{\frac{1}{6}})$.
11. Find the degree and a basis of $\mathbb{Q}(\sqrt{2}, \sqrt{3}, \sqrt{18})$ over $\mathbb{Q}$.

Unit II

12. Prove that the set of all automorphisms of a field E is a group under function composition.

13. If E is an algebraic extension of a field F, then prove that two algebraic closures $\overline{F}$ and $\overline{E}$ of F and E respectively are isomorphic.

14. If E is a finite extension of F, then prove that {E : F} divides [E : F].

Unit III

15. Let K be a finite extension of degree n of a finite field F of p^r elements. Prove that G(K/F) is cyclic of order n.

16. Let F be a field of characteristic zero, and let $F \leq E \leq K \leq \overline{F}$, where E is a normal extesion of F and K is an extension of F by radicals. Prove that G(E/F) is a solvable group.

17. Is the polynomial $x^5 - 1$ is solvable by radicals over $\mathbb{Q}$? Justify your answer.

(6 × 2 = 12 weightage)

Part C

Answer any *two* questions. Each carries 5 **weightage**.

18. (a) Let E be a simple extension F(α) of a field F, and let α be algebraic over F. Let the degree of irr(α, F) be $n \geq 1$. Show that every element β of E = F(α) can be uniquely expressed in the form $\beta = b_0 + b_1\alpha + ... + b_{n-1}\alpha^{n-1}$, where the b_i are in F.

    (b) Prove the existence of finite field $\mathbf{GF}(p^n)$ for every prime power p^n.

19. (a) State and prove Conjugation Isomorphism Theorem.

    (b) Prove that complex zeros of polynomials with real coefficients occur in conjugate pairs.

20. (a) What is the order of $G(\mathbb{Q}(\sqrt[3]{2})/\mathbb{Q})$?

    (b) Prove that every finite field is perfect.

21. Let K be the splitting field of $x^4 + 1$ over $\mathbb{Q}$.

    (a) Prove that:

        i. Show that $[K : \mathbb{Q}] = 4$

        ii. $G(K/\mathbb{Q})$ is isomorhic to Kline 4-group.

    (b) Find an intermediate field E with $\mathbb{Q} \leq E \leq K$ such that $[E : \mathbb{Q}] = 2$.

(2 × 5 = 10 weightage)