D2AMT2202 (2 Pages) Name......................

Reg.No...................

SECOND SEMESTER M.Sc. DEGREE EXAMINATION, APRIL 2023

(Regular/Improvement/Supplementary)

MATHEMATICS

FMTH2C07 - REAL ANALYSIS II

Time: 3 Hours **Maximum weightage: 30**

Part A

Answer *all* questions. Each carries 1 weightage.

1. Show that the translate of a measurable set is measurable.
2. Show that a monotone function that is defined on an interval is measurable.
3. Give an example for a bounded function on a closed and bounded interval which is Lebesgue integrable but not Riemann integrable.
4. Let $\{u_n\}$ be a sequence of nonnegative measurable functions on E and let $f = \sum_{n=1}^{\infty} u_n$ pointwise a.e. on E. Show that $\int_E f = \sum_{n=1}^{\infty} \int_E u_n$.
5. Let f be integrable over E. Show that for each $\epsilon > 0$ there is a set of finite measure E_0 for which $\int_{E \sim E_0} |f| < \epsilon$.
6. Show that an increasing function on a closed and bounded interval $[a, b]$ is of bounded variation on $[a, b]$.
7. Show that a Lipschitz on a closed and bounded interval $[a, b]$ is absolutely continuous on $[a, b]$.
8. Show that every Cauchy sequence has a rapidly Cauchy subsequence.

(8x1= 8 weightage)

Part B

Answer any *two* questions from each unit. Each carries 2 weightage.

Unit 1

9. Show that the outer measure is countably subadditive.
10. State and prove the Borel-Cantelli lemma.
11. Let f be a real-valued measurable function on E. Show that for each $\epsilon > 0$ there is a continuous function g on $\mathbb{R}$ and a closed set F contained in E for which $f = g$ on F and $m(E \sim F) < \epsilon$.

(P.T.O.)

Unit 2

12. State and prove Chebychev's inequality.

13. Let E be of finite measure. Suppose the sequence of functions $\{f_n\}$ is uniformly integrable over E and $\{f_n\} \to f$ pointwise a.e. on E. Show that f is integrable over E and $\lim_{n\to\infty} \int_E f_n = \int_E f$.

14. Show that if $\{f_n\} \to f$ in measure on E, then there is a subsequence $\{f_{n_k}\}$ that converges pointwise a.e. on E to f.

Unit 3

15. Show that a monotone function on an open interval is continuous except possibly at a countable number of points.

16. Show that a function f on a closed and bounded interval $[a, b]$ is absolutely continuous on $[a, b]$ if and only if it is an indefinite integral over $[a, b]$.

17. State and prove the Riesz-Fischer theorem.

(6x2= 12 weightage)

Part C

Answer any *two* questions. Each carries 5 weightage.

18. (a) Define a σ-algebra.

    (b) Show that the collection $\mathcal{M}$ of measurable sets is a σ-algebra.

19. (a) Let $\{f_n\}$ be a sequence of measurable functions on a measurable set E that converges pointwise a.e. on E to the function f. Show that f is measurable.

    (b) Show that an extended real-valued function f on a measurable set E is measurable if and only if there is a sequence $\{\varphi_n\}$ of simple functions on E which converges pointwise on E to f and has the property that $|\varphi_n| \leq |f|$ on E for all n.

20. Let f and g be bounded measurable functions on a set of finite measure E.

    (a) Show that for any real numbers α and β, $\int_E(\alpha f + \beta g) = \alpha \int_E f + \beta \int_E g$.

    (b) Show that if $f \leq g$ on E, then $\int_E f \leq \int_E g$.

21. Show that a monotone function defined on an open interval (a, b) is differentiable almost everywhere on (a, b).

(2x5= 10 weightage)