**D2AMT2201** **Name............................**

**Reg. No. ........................**

**SECOND SEMESTER M.Sc. DEGREE EXAMINATION, APRIL 2023**
**(Regular/Improvement/Supplementary)**
**MATHEMATICS**
**FMTH2C06: GALOIS THEORY**

**Time : 3 Hours** **Maximum Weightage: 30**

## Part A

**Answer all questions. Each carries 1 weightage**

1. Show that $\sqrt{1+\sqrt[3]{2}}$ is algebraic over $\mathbb{Q}$.
2. Is $\mathbb{C}$ a simple extension over $\mathbb{R}$? Justify your answer.
3. Find the number of primitive 18th roots of unity in GF(19).
4. Let $\sigma : \mathbb{Q}(\sqrt{2}) \to \mathbb{Q}(\sqrt{2})$ be defined by $\sigma(a+b\sqrt{2}) = 1 + b\sqrt{2}$, where $a, b \in \mathbb{Q}$. Verify whether $\sigma$ is an automorphism of $\mathbb{Q}(\sqrt{2})$.
5. What is the order of $\mathrm{G}(\mathbb{Q}(\sqrt[3]{2}, i\sqrt{3})/\mathbb{Q}(\sqrt[3]{2}))$?
6. Define separable extension of a field. Give a separable extension of $\mathbb{Q}$.
7. Is regular 18-gon is constructible? Justify your answer.
8. Define symmetric function in three variables $y_1$, $y_2$, $y_3$.

**(8 $\times$ 1 = 8 weightage)**

## Part B

**Answer any two questions from each unit. Each carries 2 weightage**

### Unit I

9. Prove that trisecting the angle is impossible.
10. Let $\alpha$ be a zero of $x^2+x+1 \in Z_2[x]$. Show that there exist a field $\mathbb{Z}_2(\alpha)$ of four elements.

**(P.T.O.)**

11. Show that a field $\mathbf{F}$ is algebraically closed if and only if every non constant polynomial in $\mathbf{F}[x]$ factors into linear factors in $\mathbf{F}[x]$ .

## Unit II

12. Prove that complex zeros of polynomials with real coefficients occur in conjugate pairs.

13. Determine the spliting field and its degree over $\mathbb{Q}$ of $x^4 + x^2 + 1$.

14. Prove that every finite field is perfect.

## Unit III

15. Prove that $x^5 - 1$ is solvable by radicals over $\mathbb{Q}$.

16. Prove that the Galois group of the $p^{th}$ cyclotomic extension of $\mathbb{Q}$ for a prime $p$ is cyclic of order $p - 1$.

17. Let $\mathbf{F}$ be a field of characteristic zero, and let $a \in \mathbf{F}$. Prove that if $\mathbf{K}$ is the splitting field of $x^n - a$ over $\mathbf{F}$, then $\mathrm{G}(\mathbf{K}/\mathbf{F})$ is a solvable group.

**(6 × 2 = 12 weightage)**

## Part C

**Answer any two questions. Each carries 5 weightage**

18. (a) Show that if $\mathbf{E}$ is finite extension field of a field $\mathbf{F}$, and $\mathbf{K}$ is a finite extension field of $\mathbf{E}$, then $\mathbf{K}$ is a finite extension of $\mathbf{F}$, and $[\mathbf{K} : \mathbf{F}] = [\mathbf{K} : \mathbf{E}][\mathbf{E} : \mathbf{F}]$.

    (b) Show that if $\mathbf{E}$ is a finite extension of $\mathbf{F}$, then $\{\mathbf{E} : \mathbf{F}\}$ divides $[\mathbf{E} : \mathbf{F}]$.

19. (a) Let $p$ be a prime and let $n \in \mathbb{Z}^+$. If $\mathbf{E}$ and $\mathbf{E}'$ are fields of order $p^n$, then show that $\mathbf{E} \simeq \mathbf{E}'$.

    (b) Let $\mathbf{F}$ be a field of characteristic $p$. Prove that the map $\sigma_p : \mathbf{F} \to \mathbf{F}$ defined by $\sigma_p(a) = a^p$ is an automorphism. Also, prove that $\mathbf{F}_{\{\sigma_p\}} = \mathbb{Z}_p$.

20. State and prove Isomorphism Extension Theorem.

21. Let $\mathbf{K}$ be a finite normal extension of a field $\mathbf{F}$ with Galois group $\mathrm{G}(\mathbf{K}/\mathbf{F})$. For each intermediate field $\mathbf{E}$ with $\mathbf{F} \leq \mathbf{E} \leq \mathbf{K}$. Let $\lambda(\mathbf{E}) = G(\mathbf{K}/\mathbf{E})$. Prove that

    (a) The fixed field of $\lambda(\mathbf{E})$ in $\mathbf{K}$ is $\mathbf{E}$.

    (b) $\lambda$ is one to one.

    (c) For $\mathbf{H} \leq \mathrm{G}(\mathbf{K}/\mathbf{F})$, $\lambda(\mathbf{K_H}) = \mathbf{H}$.

    (d) $\mathbf{E}$ is a normal extension of $\mathbf{F}$ if and only if $\lambda(\mathbf{E})$ is a normal subgroup of $\mathrm{G}(\mathbf{K}/\mathbf{F})$.

**(2 × 5 = 10 weightage)**