D2AMT2104

Reg.No.........................

Name: ....

SECOND SEMESTER M.Sc. DEGREE EXAMINATION, APRIL 2022
(Regular/Improvement/Supplementary)
MATHEMATICS
FMTH2C09: ODE AND CALCULAS OF VARIATIONS

Time: 3 Hours **Maximum Weightage: 30**

Part A: Answer all questions. Each carries 1 weightage.

1. **Find the indicial equation and its roots for the equation:** $x^3y^{''} + (cos2x - 1)y^{'} + 2xy = 0$.
2. **Show that** $\sin x = x\left[\lim_{a\longrightarrow\infty} F\left(a, a, \frac{3}{2}, \frac{-x^2}{4a^2}\right)\right]$
3. **Transform the equation** $(x^2-1)y^{''}+(5x+4)y^{'}+y = 0$ **into a Gauss's Hypergeometric equation.**
4. **Define gamma function and show that** $\Gamma(p+1) = p!$, **where** p **is a positive integer.**
5. **Show that** $\frac{d}{dx}\left[xJ_1(x)\right] = xJ_0(x)$
6. **Describe the phase portrait of the system** $\begin{cases} \frac{dx}{dt} = 1 \\ \frac{dy}{dt} = 2 \end{cases}$
7. **State Picard's theorem.**
8. **Find the normal form of Bessel's equation:** $x^2y^{''} + xy^{'} + (x^2 - p^2)y = 0$.

(8 x 1 = 8 weightage)

Part B: Answer any two questions from each unit. Each carries 2 weightage.

Unit I

9. **Show that** $\tan x = x + \frac{1}{3}x^3 + \frac{1}{3}x^5 + ...$ **by solving the equation** $y^{'} = 1 + y^2$; $y(0) = 0$ **in two ways.**
10. **Show that Gauss's Hypergeometric equation** $x(1-x)y^{''}+\left[c-(a+b+1)x\right]y^{'}-aby = 0$ **has precisely three singular points.**

11. **Determine the nature of the point $x = \infty$ for the Legendre's equation** $(1-x^2)y'' - 2xy' + p(p+1)y = 0$ **,where p is a constant.**

Unit II

12. **If** $f(x) = \begin{cases} 1, & 0 \le x < 2 \\ \frac{1}{2}, & x = \frac{1}{2} \\ 0, & \frac{1}{2} < x \le 1 \end{cases}$ **show that** $f(x) = \sum_{n=1}^{\infty} \frac{J_1(\frac{\lambda_n}{2})}{\lambda_n J_1(\lambda)^2} J_0(\lambda_n x)$ **where** λ_n **are the positive zeros of** $J_0(x)$**.**

13. **Find the general solution of the system** $\begin{cases} \frac{dx}{dt} = x + y \\ \frac{dy}{dt} = 4x - 2y \end{cases}$

14. **Consider the initial value problem** $y' = x + y$, $y(0) = 1$ **starting with** $y_0(x) = 1$, **applying Picard's method to calculate** $y_1(x), y_2(x), y_3(x)$ **and compare these result with the exact solution.**

Unit III

15. **Verify that** $(0,0)$ **is a simple critical of the system** $\begin{cases} \frac{dx}{dt} = x + y - 2xy \\ \frac{dy}{dt} = -2x + y + 3y^2 \end{cases}$ **and determine it's nature and stability properties.**

16. **Let** $y(x)$ **and** $z(x)$ **respectively be nontrivial solutions of** $y'' + q(x)y = 0$ **and** $z'' + r(x)z = 0$ **where** $q(x)$ **and** $r(x)$ **are positive functions such that** $q(x) > r(x)$**. Show that** $y(x)$ **vanishes at least once between any two successive zeros of** $z(x)$**.**

17. **A curve in the first quadrant joins (0,0) and (1,0) and has a given area beneath it. Show that the shortest such curve is an arc of a circle.**

(6 x 2 = 12 weightage)

Part C: Answer any two questions. Each carries 5 weightage .

18. **a) Find the general solution of the equation**

$$(1+x^2)y'' + 2xy' - 2y = 0.$$

b) Show that the equation $x^2y'' + xy' + (x^2-1)y = 0$ **has only one Frobenius series solution. Find it. How can the general solution be found? Give an outline.**

19. **a) State and prove orthogonality property for Bessel functions.**

b) Show that $\frac{2p}{x} J_p(x) = J_{p-1}(x) + J_{p+1}(x)$**.**

20. **a) Determine the nature and stability properties of the critical point** $(0,0)$ **for the system:** $\begin{cases} \frac{dx}{dt} = -4x - y \\ \frac{dy}{dt} = x - 2y \end{cases}$

b) State and prove Liapunov's theorem.

21. **a) Let** $f(x,y)$ **be a continuous function that satisfies a Lipschits condition** $|f(x,y_1) - f(x,y_2)| \leq k\,|y_1 - y_2|$ **on a strip defined by** $a \leq x \leq b$ **and** $-\infty < y < \infty$**. If** (x_0, y_0) **is any point on the strip, then prove that the initial value problem** $y^{'} = f(x,y), y(0) = y_0$ **has one and only one solution** $y = y(x)$ **on the interval** $a \leq x \leq b$**.**

b) Show that $f(x,y) = x^2\,|y|$ **satisfies a Lipschitz condition on any rectangle** $|x| \leq 1$ **and** $|y| \leq 1$

(2 x 5 = 10 weightage)