D2AMT2102 (2 Pages) Name..............................
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SECOND SEMESTER M.Sc. DEGREE EXAMINATION, APRIL 2022
(Regular/Improvement/Supplementary)
MATHEMATICS
FMTH2C07: REAL ANALYSIS II

Time: 3 Hours **Maximum Weightage: 30**

Part A: Answer all questions. Each carries 1 weightage.

1. If A is a finite set, then prove that A has measure zero.
2. Give an example of a non-measurable function.
3. Let $f : [1, 5] \to R$ be defined by $f(x) = x^2 + 4x + 4$. Is f measurable? Justify your answer.
4. Evaluate $\int_E f$ where $E = [0, 3]$ and

$$f(x) = \begin{cases} 1, & \text{if } 0 \leq x < 1 \\ 3, & \text{if } 1 \leq x < 2 \\ 1, & \text{if } 2 \leq x \leq 3. \end{cases}$$

5. Define functions of bounded variation. Give an example.
6. Prove that a measurable function can be expressed as the difference of two non-negative functions.
7. Every essentially bounded function is bounded. True or false? Substantiate your claim.
8. Define Banach space and give an example.

(8 x 1 = 8 weightage)

Part B: Answer any two questions from each unit. Each carries 2 weightage.

Unit 1

9. If $\{A_k\}_{k=1}^{\infty}$ is an ascending collection of measurable sets, then prove that

$$m(\bigcup_{k=1}^{\infty} A_k) = \lim_{k\to\infty} m(A_k)$$

10. Is composition of two measurable functions measurable? Justify your answer.
11. Let f be a simple function defined on E. Then prove that for each $\epsilon > 0$, there is a closed set F contained in E for which $m(E \sim F) < \epsilon$.

Unit 2

12. Define integrable function. Prove that sum of two integrable functions is integrable.

13. Prove that

$$\int_E f = \sum_{n=1}^{\infty} \int_{E_n} f$$

where f is integrable function and $\{E_n\}$ a disjoint countable collection of measurable subsets of E whose union is E.

14. Let f be a non negative function. Show that $\int_E f = 0$ if and only if $f = 0$ a.e. on E.

Unit 3

15. Let f be an increasing function defined on $[a, b]$. Then f' is integrable over $[a, b]$ and $\int_a^b f' = f(b) - f(a)$.

16. Define absolutely continuous functions. Give an example of an absolutely continuous function which is not Lipschitz.

17. Prove that a cauchy sequence in a normed linear space converges if it has a convergent subsequence.

(6 x 2 = 12 weightage)

Part C: Answer any two questions. Each carries 5 weightage.

18. (a) Prove that any set E of real numbers with positive outer measure contains a subset that fails to be measurable.

    (b) Prove that there exists disjoint set of real numbers A and B for which

$$\mathbf{m}^*(\mathbf{A} \cup \mathbf{B}) < \mathbf{m}^*(\mathbf{A}) + \mathbf{m}^*(\mathbf{B}).$$

19. State and prove Lebesgue Dominated convergence theorem.

20. (a) Prove that a monotone function on (a, b) has only a countable number of discontinuities.

    (b) Let C be a countable subset of the open interval (a, b). Is there an increasing function on (a, b) that is continuous only at points in $(\mathrm{a}, \mathrm{b}) \sim \mathrm{C}$?

21. State and prove Lusin's theorem.

(2 x 5 = 10 weightage)