D2AMT2101 **Reg.No..........................**

Name: ....

SECOND SEMESTER M.Sc. DEGREE EXAMINATION, APRIL 2022
(Regular/Improvement/Supplementary)
MATHEMATICS
FMTH2C06: GALOIS THEORY

Time : 3 Hours **Maximum Weightage: 30**

Part A

Answer all questions. Each carries 1 weightage.

1. If $\alpha = \sqrt{\pi}$, then determine deg(α, Q(π)).
2. Find a basis of Q($\sqrt{2}$, $\sqrt{6}$) over Q($\sqrt{3}$).
3. Does there exist a field of 4096 elements? Justify your answer.
4. Find all conjugates of $\sqrt{2} + i$.
5. Write the splitting field of $\{x^2 - 2,\ x^2 - 3\ \}$ over Q.
6. Show that if $\alpha,\ \beta \in \overline{F}$ are both seperable over a field F, then both $\alpha + \beta$ and $\alpha\beta$ are seperable over F.
7. Is it true that the regular 18-gon constructible? Justify your answer.
8. Define the n^{th} cyclotomic polynomial over a field F.

(8 × 1 = 8 weightage)

Part B

Answer any two questions from each unit. Each carries 2 weightage.

Unit I

9. Prove that a finite extension is an algebraic extension.
10. Let E be an algebraic extension of a field F and $\alpha \in$ E. Let irr(α, F) denote the irreducible polynomial for α over F. Prove that:

    (a) If α is algebraic over F, then F(α) is isomorphic to F[x]/$\langle$irr(α, F)$\rangle$.
    (b) If α is transcendental over F then F(α) is isomorphic to the field of rational functions over F.
11. Show that there exists an angle that cannot be trisected with a straightedge and a compass.

(P.T.O.)

Unit II

12. Let F be a field of characteristic p. Prove that the map σ_p : F $\rightarrow$ F defined by $\sigma_p(a) = a^p$ is an automorphism. Also prove that $F_{\{\sigma_p\}} \simeq Z_p$.

13. Show that if [E : F] = 2, then E is a spliting field over F.

14. Prove that every field of characteristic zero is perfect.

Unit III

15. Find the Galois group of Q($\sqrt{2}, \sqrt{3}$) over Q.

16. Prove that the Galois group of the p^{th} cyclotomic extension of Q for a prime p is cyclic of order $p - 1$.

17. Let F be a field of charectristic 0, and let a $\in$ F. If K is the spliting field of $x^n - a$ over F and F contains all the n^{th} roots of unity, then prove that that G(K/F) is a solvable group.

(6 $\times$ 2 = 12 weightage)

Part C

Answer any two questions. Each carries 5 weightage.

18. (a) Let E be a finite extension of degree n over a finite field F. If F has q elements then prove that E has q^n elements.

    (b) Show that every finite extension of a finite field is a simple extension.

    (c) Find all primitive 10^{th} roots of unity in Z_{11}.

19. State and Prove Isomorphism Extension Theorem.

20. (a) Define the index $\{E : F\}$ of E over F.

    (b) Let E be finite extension of a field F. Let σ be an isomorphism of F onto a field F' and let $\overline{F'}$ be an algebraic closure of F'. Prove that the number of extensions of σ to an isomorphism τ of E onto a subfield of $\overline{F'}$ is finite and independent of F', $\overline{F'}$ and σ.

21. Let K be a finite normal extension of a field F, and let F $\leq$ E $\leq$ K $\leq$ $\overline{F}$. Prove that:

    (a) K is a finite normal extension of E.

    (b) For σ, τ $\in$ G(K/F), the restrictions $\sigma|E = \tau|E$ if and only if σ and τ are in the same left coset of G(K/E) in G(K/F).

    (c) The fixed field of G(K/E) in K is E.

(2 $\times$ 5 = 10 weightage)