D2AST2003 (2 PAGES) Reg. No. ···············

Name:··················

SECOND SEMESTER M.Sc. DEGREE EXAMINATION, APRIL 2021
STATISTICS
FMST2C08: PROBABILITY THEORY

Time: Three Hours **Maximum Weightage: 30**

Part A: All questions can be answered. Each carries two weightage. (Ceiling 6 weightage).

1. Define random variable. Prove that sum of two random variables on a probability space is again a random variable on that probability space.
2. Prove that the distribution function of a random variable is right continuous.
3. Does convergence in distribution implies convergence in probability? Justify your answer.
4. Examine whether WLLN holds for a sequence $\{X_n\}$ of independent random variables, where

$$P(X_n = 2^n) = \frac{1}{2} = P(X_n = -2^n), \quad n \geq 1$$

5. Define characteristic function. Is $\phi(t) = \cos t,\ t \in \mathcal{R}$; a characteristic function? Justify your answer.
6. What is central limit problem? Define any one form.
7. State Radan-Nikodym theorem. What is its significance in probability theory?

Part B: All questions can be answered. Each carries four weightage. (Ceiling 12 weightage).

8. If F is a distribution function prove that F can be uniquely decomposed as:

$$F = \alpha F_d + (1-\alpha)F_c,\ 0 < \alpha < 1,$$

where F_d is a discrete distribution function and F_c a continuous distribution function.

9. Define tail events. State and prove Kolmogorov 0 - 1 law.

(PTO)

10. Define weak convergence of a sequence of distribution functions. Let $\{F_n\}$ be a sequence of distribution functions such that

$$F_n(x) = \begin{cases} 0, & \text{if} \quad x \leq 0, \\ x^n, & \text{if} \quad 0 < x \leq 1, \\ 1, & \text{if} \quad x > 1 \end{cases}$$

Examine whether $\{F_n\}$ converges weakly. Identify the limit if it exists.

11. State and prove necessary and sufficient condition for almost sure convergence of a series of independent random variables as suggested by Kolmogorov.

12. State and prove the inversion theorem on characteristic functions.

13. State Liapunov's form of central limit theorem. Examine whether Liapunov's conditions are satisfied by the sequence X_n, where

$$P(X_n = n) = \frac{1}{2\sqrt{n}} = P(X_n = -n)$$

$$P(X_n = 0) = 1 - \frac{1}{\sqrt{n}} \quad n = 1, 2, \ldots$$

14. Define conditional expectation. Discuss the smoothing properties of conditional expectation.

Part C: All questions can be answered. Each carries six weightage. (Ceiling 12 weightage).

15. a) Define probability measure induced by a random variable X. Show that it satisfies all the axioms of probability.

    b) Describe independence of random variables. If X and Y are independent random variables and $g : \mathcal{R} \to \mathcal{R}$ is a continuous function, prove that $g(X)$ and $g(Y)$ are independent.

16. a) Define almost sure convergence and convergence in probability. Prove that almost sure convergence implies convergence in probability.

    b) State and prove Kolmogrov's SLLN.

17. a) State and prove Helly-Bray lemma.

    b) Examine whether central limit theorem holds for the sequence $\{X_n\}$ of independent random variables with probability mass function:

$$P(X_n = n) = \frac{1}{2n^3} = P(X_n = -n)$$

$$P(X_n = 0) = 1 - \frac{1}{n^3} \quad n = 1, 2, \ldots$$

18. a) Prove that $E(X + Y|\mathcal{B}) = E(X|\mathcal{B}) + E(Y|\mathcal{B})$

    b) Let $\{X_n\}$ be a martingale and g(.) be a convex function on $\mathcal{R}$. Show that $\{g(X_n)\}$ is a sub-martingale provided $E(g(X_n)) < \infty$ for $n \geq 1$.