D2AMT2002 (2 Pages) Name:..............................
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SECOND SEMESTER M.Sc. DEGREE EXAMINATION, APRIL 2021
MATHEMATICS
FMTH2C07: REAL ANALYSIS II

Time: 3 Hours **Maximum Weightage: 30**

Part A: All questions can be answered. Each carries one weightage. (Ceiling 6 weightage)

1. Prove that a countable set has measure zero.
2. Prove that characteristic function on A, $\chi_{\mathbf{A}}$ is measurable if and only if A is measurable.
3. If $\mathrm{m}^*(\mathrm{A}) = 0$, then prove that $\mathrm{m}^*(\mathrm{A} \cup \mathrm{B}) = \mathrm{m}^*(\mathrm{B})$.
4. Give an example of a closed uncountable set with measure zero.
5. Give an example of a Lebesgue integrable function which is not Riemann integrable.
6. Let f and g be integrable over a measurable set E. If $f \leq g$, then prove that
$$\int_E f \leq \int_E g$$
7. Define absolutely continuous functions.
8. Every essentially bounded function is bounded. True or false? Substantiate your claim.

Part B: All questions can be answered. Each carries two weightage. (Ceiling of 12 weightage).

9. Let f be a function defined on a measurable set E. Then f is measurable if and only if for each open set O, the inverse image of O under f, $f^{-1}(O)$ is measurable.
10. (a) Show that a monotone function that is defined on an interval is measurable.

    (b) Let f and g are two measurable functions on E that are finite a.e. on E, then prove that fg is measurable on E.

11. Let $\{f_n\}$ be a sequence of measurable functions on E that converges pointwise a.e. on E to the function f. Then f is measurable.

12. (a) Let f be a non-negative measurable function on E. Then prove that $\int_E f = 0$ if and only if $f = 0$ a.e. on E.

    (b) Prove that a non-negative integrable function on a measurable set E is finite a.e. on E.

13. If f and g are integrable on a measurable set E, then prove that $f + g$ is integrable over E and $\int_E (f + g) = \int_E f + \int_E g$

14. State and prove the Monotone convergence theorem.

15. Let f on R be given by $f(x) = \begin{cases} x\sin(1/x) & x \neq 0 \\ 0 & x = 0 \end{cases}$ Find the upper and lower derivatives of f at $x = 0$.

16. Define functions of bounded variation. Prove that a Lipschitz function on $[a, b]$ is of bounded variation.

17. State and prove Minkowski's inequality.

Part C: All questions can be answered. Each carries six weightage. (Ceiling 12 weightage).

18. (a) Prove that every Borel set is measurable.

    (b) State and prove simple approximation theorem.

19. (a) State and prove bounded convergence theorem.

    (b) Let f be a measurable function on E. Suppose there is a non-negative function g that is integrable over E and $|\mathrm{f}| \leq \mathrm{g}$ on E. Then prove that

$$\left|\int_{\mathbf{E}} \mathbf{f}\right| \leq \int_{\mathbf{E}} |\mathbf{f}|.$$

20. (a) Prove that a function F is an indefinite integral over $[\mathrm{a}, \mathrm{b}]$ if and only if it is absolutely continuous on $[\mathrm{a}, \mathrm{b}]$.

    (b) If f is an increasing function defined on $[a, b]$ and

$$\int_{\mathbf{a}}^{\mathbf{b}} \mathbf{f}' = \mathbf{f}(\mathbf{b}) - \mathbf{f}(\mathbf{a}).$$

    Then prove that f is absolutely continuous.

21. Assume E has a finite measure. Let $\{f_n\}$ be a sequence of measurable functions on E that converges pointwise a.e. on E to f and f is finite a.e. on E. Then prove that $f_n \to f$ in measure on E. Also prove by an example that converse of this result is not true.