D2AMT2001 (2 PAGES) Reg. No. ···············

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SECOND SEMESTER M.Sc. DEGREE EXAMINATION, APRIL 2021

MATHEMATICS

FMTH2C06: GALOIS THEORY

Time: Three Hours **Maximum Weightage: 30**

Part A: All questions can be answered. Each carries one weightage.

(Ceiling 6 weightage).

1. Let f(x) be irreducible in F[x]. Then show that all zeros of f(x) in $\overline{F}$ have same multiplicity.
2. Find the splitting field of $X^5 - 1$ over $\mathbb{Q}$.
3. Prove that doubling a cube is impossible.
4. Describe the group $G(\mathbb{Q}(\sqrt{2}, \sqrt{3})/\mathbb{Q})$.
5. Show that $\mathbb{Q}(\sqrt[3]{2})$ has only identity automorphism.
6. Find the degree and basis of $\mathbb{Q}(2^{\frac{1}{2}}, 2^{\frac{1}{3}})$ over $\mathbb{Q}$.
7. Prove that a finite extension of a field F is an algebraic extension of F.
8. Is $\alpha = \pi^2$ is algebraic over $\mathbb{Q}(\pi^3)$. Justify your answer.

Part B: All questions can be answered. Each carries two weightage.

(Ceiling 12 weightage).

9. a) Prove that there exists an angle that cannot be trisected.

   b) Show that a regular 9-gon is not constructible.
10. Prove in detail that $\mathbb{Q}(\sqrt{3}, \sqrt{7}) = \mathbb{Q}(\sqrt{3} + \sqrt{7})$.
11. Let E be a simple extension $F(\alpha)$ of a field F, and let α algebraic over F. Let the degree of $irr(\alpha, F)$ be $n \geq 1$. Then prove that every element β in E can be uniquely expressed as a linear combination of $\{1, \alpha, \alpha^2, ..., \alpha^{n-1}\}$ with co-efficients from F.

12. Prove that a finite extension of a field of characteristic zero is a simple extension.

13. a) State and prove conjugation isomorphism theorem.

    b) Prove that complex zeros of a polynomial with real coefficients occur in conjugate pairs.

14. Show that if [E:F] = 2 , Then show that E is a splitting field over F.

15. Find $\phi_5(x)$ over $\mathbb{Q}$.

16. a) Check whether regular 60-gon in constructible.

    b) Define nth cylclotomic polynomial and nth cylclotomic extension over a field F.

17. Prove that the Galois group of the pth cyclotomic extension of $\mathbb{Q}$ is cyclic of order $p-1$.

Part C: All questions can be answered. Each carries six weightage.

(Ceiling 12 weightage).

18. a) State and prove Kronecker's theorem.

    b) Construct a field of four elements.

    c) Prove that $\mathbb{R}[x]/ < x^2 + 1 >$ is isomorphic to $\mathbb{C}$.

19. State and prove isomorphism extension theorem.

20. a) Prove that a finite field $GF(p^n)$ exists for every prime power p^n.

    b) Let P be a prime and n be a posiive integer, Prove that if E and E' are fields of order p^n then they are isomorphic.

21. Let $y_1, y_2, ..., y_5$ be independent transcendental real numbers over $\mathbb{Q}$. Prove that the polynomial $f(X) = \prod_{i=1}^{5}(x - y_i)$ is not solvable by radicals over $F = \mathbb{Q}(s_1, s_2, ..., s_5)$, where s_i is the ith elementary symmetric function in $y_1, y_2, ..., y_5$.